

CHEVALLEY–MONK FORMULAS FOR BOW VARIETIES

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ABSTRACT. We prove a formula for the multiplication of equivariant first Chern classes of tautological bundles of type A bow varieties with respect to the stable envelope basis. This formula naturally generalizes the classical Chevalley–Monk formula and can be formulated in terms of creating crossings of skein type diagrams that label the stable envelope basis.

CONTENTS

1. Introduction	1
2. Bow varieties	3
3. Cotangent bundles of partial flag varieties as bow varieties	13
4. Stable envelopes	17
5. Stable bases in the separated case	20
6. Stable bases of cotangent bundles of flag varieties	23
7. Symmetric group calculus for bow varieties	28
8. Equivariant resolution theorem	33
9. Chevalley–Monk formulas in the separated case	39
10. Chevalley–Monk formulas in the general case	49
References	53

1. INTRODUCTION

By introducing the theory of *stable envelopes*, Maulik and Okounkov provided in [MO19] a way to assign Hopf algebras (called *quantum groups*) to a large family of symplectic varieties with appropriate torus action. Given such a variety X then stable envelopes provide families of bases of the localized torus equivariant cohomology of X which are uniquely characterized by stability conditions which are similar to the stability conditions from equivariant Schubert calculus. The base change matrices of these bases give solutions of the Yang–Baxter equation and the corresponding quantum group is defined via the usual FRT-construction [FRT88]. By construction, this quantum group naturally acts on the equivariant cohomology of X and contains all operators of multiplication and (more generally) quantum multiplication with equivariant cohomology classes.

The main focus of the work [MO19] lies on the case where X is a Nakajima quiver variety, which includes the special case where X is the cotangent bundle of a partial flag variety, the Hilbert schemes of points in the complex plane or (more generally) a moduli spaces of framed torsion sheaves. In their study, the first Chern classes of tautological bundles are of great significance as they are well behaved with respect to the stable envelopes basis.

These operators can be described in terms of well-known Hamiltonians of integrable models in specific examples. For instance, in the case of Hilbert schemes of points in the complex plane, the operator of multiplication with the first Chern class of the tautological bundle coincides with the Calogero–Moser–Sutherland Hamiltonian (see e.g. [CG03]), whereas the operator of quantum multiplication yields a deformation of this Hamiltonian, see [OP10]. In the case of cotangent bundles of partial flag varieties, the operators of multiplication and quantum multiplication with first Chern classes of tautological bundles are identified with the dynamical Hamiltonians for the XXX model for general linear groups as defined by Tarasov and Varchenko, see [TV00], [TV05], [TV14] and [RTV15].

In this article, we state and prove a formula for the multiplication of first Chern classes of tautological bundles with respect to the stable envelope basis for bow varieties. These varieties are a very general class of varieties which again are defined in terms of quiver representations and some Hamiltonian reductions construction extending the class of type A Nakajima quiver varieties.

Bow varieties were defined by Nakajima and Takayama in [NT17]. They form a rich family of smooth symplectic varieties which fits into the framework of Maulik and Okounkov’s theory. They have their origins in theoretical physics, where they appear as certain instanton moduli spaces, see [Che09], [Che10] and [Che11]. In particular, they play a remarkable role in the context of 3d mirror symmetry which is a phenomenon from theoretical physics which relates different field theories, see in particular [NT17].

By construction, bow varieties are naturally endowed with a family of tautological vector bundles. They also admit a torus action which scales the symplectic form and has only finite fixed point locus. The torus fixed points of bow varieties were classified by Nakajima in [Nak18, Theorem A.5] in terms of Maya diagrams or equivalently partitions. In [RS24], Rimányi and Shou give an equivalent classification of torus fixed points in terms of skein type diagrams called *tie diagrams* which naturally extend the torus fixed point combinatorics of partial flag varieties:

$$\{\text{Tie diagrams}\} \xleftrightarrow{1:1} \{\text{Torus fixed points}\}, \quad D \mapsto x_D.$$

Our main result is then an *explicit* formula for the multiplication of the tautological bundles (see Theorem 9.1 and Theorem 10.4):

Theorem (Multiplication formula). *Let $\mathcal{C}(\mathcal{D})$ be a bow variety and $(\text{Stab}(x_D))_D$ a fixed choice of stable basis and ξ_i a tautological bundle. Then, we have*

$$c_1(\xi_i) \cup \text{Stab}(x_D) = \iota_{x_D}^*(c_1(\xi_i)) \cdot \text{Stab}(x_D) + \sum_{D' \in \text{SM}_{D,i}} \text{sgn}(D, D') h \cdot \text{Stab}(x_{D'}), \quad (1.1)$$

where h is the equivariant parameter corresponding to the scaling of the symplectic form, $\text{SM}_{D,i}$ a certain set of tie diagrams which are obtained from D by resolving one crossing and $\text{sgn}(D, D')$ a explicitly computable sign.

The formula holds in the localized torus equivariant cohomology of $\mathcal{C}(\mathcal{D})$. The localization coefficients of $c_1(\xi_i)$ which appear in (1.1) can be explicitly determined using the formula from [RS24, Theorem 4.10].

The formula (1.1) generalizes the classical Chevalley–Monk formula from Schubert calculus as follows: if our bow variety is the cotangent bundle of the full flag variety F_n then the

torus fixed points are labeled by elements in the symmetric group S_n and the tautological bundles ξ_i correspond to the universal quotient bundles $\mathcal{Q}_1, \dots, \mathcal{Q}_{n-1}$. In this special case, the formula (1.1) was already proved by Su in [Su16, Theorem 3.7] and can be reformulated as

$$c_1(\mathcal{Q}_i) \cup \text{Stab}(x_w) = \iota_{x_w}^*(c_1(\mathcal{Q}_i)) \cdot \text{Stab}(x_w) + \sum_{\substack{j \leq i < k \\ \ell(wt_{k,j}) < \ell(w)}} h \cdot \text{Stab}(x_{wt_{k,j}}), \quad (1.2)$$

where $w \in S_n$, l is the Bruhat length and $t_{k,j} \in S_n$ the transposition switching k and j . Since basis elements can be interpreted as one-parameter deformations of Schubert classes, a well-known limit argument (see e.g. [AMSS23, Section 9]) proves that (1.2) implies the classical Chevalley–Monk formula in the singular cohomology of F_n ([Che94], [Mon59]):

$$c_1(\mathcal{Q}_i) \cup \mathfrak{S}_w = \sum_{\substack{j \leq i < k \\ \ell(wt_{k,j}) = \ell(w) + 1}} \mathfrak{S}_{wt_{k,j}},$$

where \mathfrak{S}_w denotes the Schubert class corresponding to the permutation w .

For the proof of (1.1), we employ the equivariant resolution theorem of Botta and Rimányi [BR23, Theorem 6.13] that provides a way to compute localization coefficients of stable envelopes of bow varieties in terms of localization coefficients of stable envelopes of cotangent bundles of partial flag varieties. Then applying the localization formula from [Su17b] allows us to express these localization coefficients in terms of symmetric group combinatorics which gives a good control over them.

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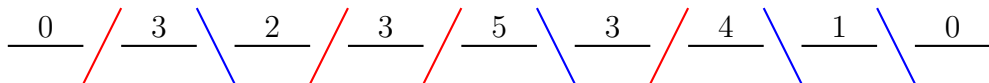
2. BOW VARIETIES

In this section, we recall the construction of bow varieties from [NT17] and state some of their important properties following [NT17] and [RS24].

Convention. If not stated otherwise, all varieties and vector spaces are over \mathbb{C} . Given a variety Y and a smooth point $y \in Y$ we denote by $T_y Y$ the tangent space of Y at y .

2.1. Brane diagrams. For the construction of bow varieties, we use the language of brane diagrams introduced by Rimányi and Shou which we now briefly recall.

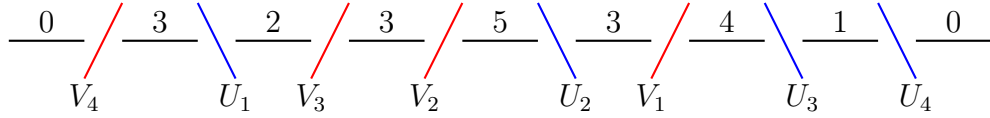
A *brane diagram* is an object like this:



That is, a brane diagram is a finite sequence of black horizontal lines drawn from left to right. Between each consecutive pair of horizontal lines there is either a blue SE–NW line \backslash or a red SW–NE line $/$. Each horizontal line X is labeled by a non-negative integer d_X where we demand that the first and the last horizontal line is labeled by 0.

Remark 2.1. In [RS24], the black horizontal lines are, motivated by string theory, called D3 branes, the blue lines D5 branes and the red lines NS5 branes. Since brane diagrams appear for us only as purely combinatorial objects, we will not use this naming, but refer to their color instead.

Given a brane diagram \mathcal{D} , we denote by $h(\mathcal{D})$, $b(\mathcal{D})$ and $r(\mathcal{D})$ the subset of black, blue and red lines. For lines Y_1, Y_2 in \mathcal{D} write $Y_1 \triangleleft Y_2$ if Y_1 is to the left of Y_2 . We denote the number of red lines by M and the individual red lines by V_1, \dots, V_M where we number the lines from *right to left*. Likewise, let N be the number of blue lines and the blue lines are denoted by U_1, \dots, U_N numbered from *left to right*. So in the above brane diagrams the colored lines are labeled like this:



The black lines are denoted by X_1, \dots, X_{M+N+1} numbered from *left to right*. So in the above example $(d_{X_1}, \dots, d_{X_9}) = (0, 3, 2, 3, 5, 3, 4, 1, 0)$. Note that our choice of labeling of red lines differs from [RS24].

The *separatedness degree* of \mathcal{D} is defined as

$$\text{sdeg}(\mathcal{D}) = |\{(U, V) \in b(\mathcal{D}) \times r(\mathcal{D}) \mid U \triangleleft V\}|. \quad (2.1)$$

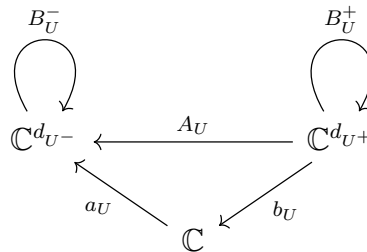
We call \mathcal{D} *separated* if $\text{sdeg}(\mathcal{D}) = 0$. So \mathcal{D} is separated if and only if it has the shape $// \dots // \backslash \dots \backslash$.

Bow varieties associated to separated brane diagrams have many convenient properties, as we will discuss in Section 5 and Section 8. There are explicit moves on brane diagrams (called *Hanany–Witten transition*) that reduce the separatedness degree by 1 which we discuss in Subsection 2.9.

2.2. Construction of bow varieties. Let \mathcal{D} be a brane diagram. The construction of its associated bow variety proceeds through several steps. At first, we assign to each blue line $U \in b(\mathcal{D})$ a smooth affine variety tri_U called *the triangle part (of U)* as follows: we define

$$\mathbb{M}_U := \text{Hom}(\mathbb{C}^{d_{U^+}}, \mathbb{C}^{d_{U^-}}) \oplus \text{End}(\mathbb{C}^{d_{U^+}}) \oplus \text{End}(\mathbb{C}^{d_{U^-}}) \oplus \text{Hom}(\mathbb{C}^{d_{U^+}}, \mathbb{C}) \oplus \text{Hom}(\mathbb{C}, \mathbb{C}^{d_{U^-}})$$

and denote the elements of \mathbb{M}_U by tuples $(A_U, B_U^+, B_U^-, a_U, b_U)$ according to the diagram:



The *triangle part* tri_U is then defined as $\text{tri}_U = \{x \in \mu^{-1}(0) \mid x \text{ satisfies (S1), (S2)}\}$, where the map $\mu: \mathbb{M}_U \rightarrow \text{Hom}(\mathbb{C}^{d_{U^+}}, \mathbb{C}^{d_{U^-}})$ is given by

$$(A_U, B_U^+, B_U^-, a_U, b_U) \mapsto B_U^- A_U - A_U B_U^+ + a_U b_U \quad (2.2)$$

and the conditions (S1) and (S2) are defined as

(S1) If $S \subset \mathbb{C}^{d_{U^+}}$ is a subspace with $B_U^+(S) \subset S$, $A_U(S) = 0$ and $b_U(S) = 0$ then $S = 0$.

(S2) If $T \subset \mathbb{C}^{d_{U^-}}$ is a subspace with $B_U^-(T) \subset T$ and $\text{Im}(A_U) + \text{Im}(a_U) \subset T$ then $T = \mathbb{C}^{d_{U^-}}$.

The following was shown in [Tak16, Proposition 2.20]:

Proposition 2.2. *The triangle part tri_U is a smooth and affine open subvariety of $\mu^{-1}(0)$.*

The next step is to define the *affine brane variety* $\widetilde{\mathcal{M}}(\mathcal{D})$ as follows:

$$\widetilde{\mathcal{M}}(\mathcal{D}) := \left(\prod_{V \in \text{r}(\mathcal{D})} \text{Hom}(\mathbb{C}^{d_{V^+}}, \mathbb{C}^{d_{V^-}}) \times \text{Hom}(\mathbb{C}^{d_{V^-}}, \mathbb{C}^{d_{V^+}}) \right) \times \left(\prod_{U \in \text{b}(\mathcal{D})} \text{tri}_U \right).$$

Denote the elements of $\text{Hom}(\mathbb{C}^{d_{V^+}}, \mathbb{C}^{d_{V^-}}) \times \text{Hom}(\mathbb{C}^{d_{V^-}}, \mathbb{C}^{d_{V^+}})$ by (C_V, D_V) and consequently elements of $\widetilde{\mathcal{M}}(\mathcal{D})$ by tuples

$$((A_U, B_U^+, B_U^-, a_U, b_U)_U, (C_V, D_V)_V).$$

There is a natural algebraic action of $\mathcal{G} := \prod_{X \in \text{h}(\mathcal{D})} \text{GL}_{d_X}$ on $\widetilde{\mathcal{M}}(\mathcal{D})$ given by

$$(g_X)_X \cdot ((A_U, B_U^+, B_U^-, a_U, b_U)_U, (C_V, D_V)_V) \\ = ((g_{U^-} A_U g_{U^+}^{-1}, g_{U^+} B_U^+ g_{U^+}^{-1}, g_{U^-} B_U^- g_{U^-}^{-1}, g_{U^-} a_U, b_U g_{U^+}^{-1})_U, (g_{V^-} C_V g_{V^+}^{-1}, g_{V^+} D_V g_{V^+}^{-1})_V).$$

As shown in [NT17] that $\widetilde{\mathcal{M}}(\mathcal{D})$ admits a \mathcal{G} -equivariant symplectic structure with moment map

$$m: \widetilde{\mathcal{M}}(\mathcal{D}) \longrightarrow \prod_{X \in \text{h}(\mathcal{D})} \text{End}(\mathbb{C}^{d_X}),$$

given for $x = ((A_U, B_U^+, B_U^-, a_U, b_U)_U, (C_V, D_V)_V)$ by

$$m(x)_X = \begin{cases} B_{X^+}^- - B_{X^-}^+ & \text{if } X^+, X^- \text{ are both blue,} \\ D_{X^-} C_{X^-} - C_{X^+} D_{X^+} & \text{if } X^+, X^- \text{ are both red,} \\ D_{X^-} C_{X^-} + B_{X^+}^- & \text{if } X^+ \text{ is blue and } X^- \text{ is red,} \\ -C_{X^+} D_{X^+} - B_{X^-}^+ & \text{if } X^+ \text{ is red and } X^- \text{ is blue.} \end{cases}$$

Definition 2.3. Fix the character $\chi: \mathcal{G} \rightarrow \mathbb{C}^*$, $(g_X)_X \mapsto \prod_{X \in M_{\mathcal{D}}} \det(g_X)$, where $M_{\mathcal{D}}$ denote the set of black lines $X \in \text{h}(\mathcal{D})$ such that X^- is red. Then, the *bow variety associated to \mathcal{D}* is defined as the G.I.T. quotient

$$\mathcal{C}(\mathcal{D}) := m^{-1}(0) //_{\chi} \mathcal{G}.$$

The following properties of $\mathcal{C}(\mathcal{D})$ were shown in [NT17, Section 2]:

Theorem 2.4. *The following holds:*

- (i) *The χ -stable locus $m^{-1}(0)^s$ equals the χ -semistable locus $m^{-1}(0)^{\text{ss}}$.*
- (ii) *$\mathcal{C}(\mathcal{D})$ is a smooth and symplectic variety.*
- (iii) *The projection $m^{-1}(0)^s \rightarrow \mathcal{C}(\mathcal{D})$ is a principal \mathcal{G} -bundle in the Zariski topology.*

Remark 2.5. The family of bow varieties contains type A Nakajima quiver varieties which can be realized as bow varieties corresponding to so called cobalanced brane diagrams, see [NT17, Section 2.6] for more details. In Section 3, we consider this realization in the special case of cotangent bundles of flag varieties.

2.3. Torus actions. As described in [NT17, Section 6.9.3], $\mathcal{C}(\mathcal{D})$ is equipped with two torus actions. One action of $\mathbb{A} = (\mathbb{C}^*)^N$ and one of $\mathbb{C}_h^* = \mathbb{C}^*$. The \mathbb{A} -action is a direct consequence of the construction which we call the *obvious action*. This action leaves the symplectic form on $\mathcal{C}(\mathcal{D})$ invariant. The \mathbb{C}_h^* -action induces a scaling of the symplectic form and hence we refer to it as the *scaling action*. In our exposition of these actions, we follow the conventions from [RS24, Section 3.1]. Denote the elements of $\mathbb{A} = (\mathbb{C}^*)^N$ by (t_1, \dots, t_N) or $(t_U)_{U \in \mathbf{b}(\mathcal{D})}$ or just $(t_U)_U$. The obvious action is given by

$$(t_U)_U \cdot [(A_U, B_U^+, B_U^-, a_U, b_U)_U, (C_V, D_V)_V] = [(A_U, B_U^+, B_U^-, a_U t_U^{-1}, t_U b_U)_U, (C_V, D_V)_V].$$

We denote the elements of \mathbb{C}_h^* by h . Then, the scaling action is given by

$$h \cdot [(A_U, B_U^+, B_U^-, a_U, b_U)_U, (C_V, D_V)_V] = [(A_U, h B_U^+, h B_U^-, a_U, h b_U)_U, (h C_V, D_V)_V].$$

The \mathbb{T} -equivariant cohomology algebra of $\mathcal{C}(\mathcal{D})$ for the large torus $\mathbb{T} := \mathbb{A} \times \mathbb{C}_h^*$ is one of the main actors of this article. Our main tool in the study of this algebra is the localization principle which allows a powerful interplay of local and global data. To apply the localization principle appropriately, we briefly recall the classification of \mathbb{T} -fixed points of bow varieties as presented in [RS24, Section 4], in the upcoming subsections. For this, we first define certain combinatorial objects that can be assigned to brane diagrams.

2.4. Tie diagrams. Given a pair of colored lines (Y_1, Y_2) in \mathcal{D} with $Y_1 < Y_2$ then we say that a black line X is *covered* by (Y_1, Y_2) if Y_1 is to the left of X and Y_2 is to the right of X .

Definition 2.6. A *tie data with underlying brane diagram* \mathcal{D} is the data of \mathcal{D} together with a set D of pairs of colored lines of \mathcal{D} such that the following holds:

- If $(Y_1, Y_2) \in D$ then $Y_1 \triangleleft Y_2$.
- If $(Y_1, Y_2) \in D$ then either Y_1 is blue and Y_2 is red or Y_1 is red and Y_2 is blue.
- For all black lines X of \mathcal{D} , the number of pairs in D covering X is equal to d_X .

Usually, we work with a fixed brane diagram \mathcal{D} , so we refer to a tie data just by the set D . The set of all tie data corresponding to \mathcal{D} is denoted by $\text{Tie}(\mathcal{D})$.

A tie data D can be visualized as follows: attach to the brane diagram \mathcal{D} dotted curves (that are called *ties*) following the conventions:

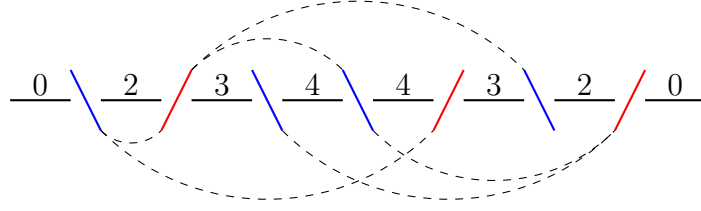
- For each pair $(Y_1, Y_2) \in D$ with Y_1 blue and Y_2 red, we draw a dotted curve below the diagram \mathcal{D} .
- For each pair $(Y_1, Y_2) \in D$ with Y_1 red and Y_2 blue, we draw a dotted curve above the diagram \mathcal{D} .

The resulting diagram is then called *tie diagram of D* . In the following, we also refer to the elements of $\text{Tie}(\mathcal{D})$ as tie diagrams.

Example 2.7. Let \mathcal{D} be the brane diagram $0 \backslash 2/3 \backslash 4/4/3 \backslash 2/0$ and D the tie data

$$D = \{(U_1, V_3), (U_1, V_2), (V_3, U_3), (V_3, U_4), (U_2, V_1), (U_3, V_1)\}.$$

Then, the visualization of D is given as follows:



2.5. Torus fixed points. In [RS24, Section 4], Rimányi and Shou give an explicit construction which assigns a \mathbb{T} -fixed point $x_D \in \mathcal{C}(\mathcal{D})$ to each tie diagram $D \in \text{Tie}(\mathcal{D})$. Based on this construction, they proved that there is a bijection

$$\text{Tie}(\mathcal{D}) \xrightarrow{\sim} \mathcal{C}(\mathcal{D})^{\mathbb{T}}, \quad D \mapsto x_D \quad (2.3)$$

which classifies the \mathbb{T} -fixed point of $\mathcal{C}(\mathcal{D})$. We will usually identify a tie diagram D with its corresponding \mathbb{T} -fixed point x_D .

The classification result (2.3) can be strengthened in the following way: for a cocharacter

$$\sigma: \mathbb{C}^* \rightarrow \mathbb{A}, \quad t \mapsto (\sigma_U(t))_U,$$

we denote by $\mathcal{C}(\mathcal{D})^\sigma$ the corresponding \mathbb{C}^* -fixed locus. If σ is *generic*, that means if $\sigma_U \neq \sigma_{U'}$ for all $U, U' \in \text{b}(\mathcal{D})$, then $\mathcal{C}(\mathcal{D})^{\mathbb{T}} = \mathcal{C}(\mathcal{D})^\sigma$, see e.g. [SW23, Theorem 4.14].

The bijection (2.3) implies that $\mathcal{C}(\mathcal{D})$ admits a \mathbb{T} -fixed point if and only if $\text{Tie}(\mathcal{D}) \neq \emptyset$. We call a brane diagram \mathcal{D} *admissible* if $\text{Tie}(\mathcal{D}) \neq \emptyset$. As the theory of stable envelopes is based on the localization principle in equivariant cohomology, we make the following assumption:

Assumption. For the remainder of this article, we assume that \mathcal{D} is admissible.

Remark 2.8. The \mathbb{T} -fixed points of bow varieties were first classified in [Nak18, Theorem A.5]. For more details on the relation between this classification and the classification from [RS24], see [Sho21, Appendix B].

2.6. Binary contingency tables. We continue with giving an equivalent definition of tie diagrams in terms of matrices with entries in $\{0, 1\}$ which satisfies convenient compatibilities as we will discuss in Subsection 2.9.

Given a brane diagram \mathcal{D} , we first assign the following invariants to \mathcal{D} :

$$r_i(\mathcal{D}) := d_{V_i^+} - d_{V_i^-} + |\{U \in \text{b}(\mathcal{D}) \mid U \triangleleft V_i\}|, \quad c_j(\mathcal{D}) := d_{U_j^-} - d_{U_j^+} + |\{V \in \text{r}(\mathcal{D}) \mid U_j \triangleleft V\}|,$$

where $i \in \{1, \dots, M\}, j \in \{1, \dots, N\}$. In addition, we set

$$R_l(\mathcal{D}) := \sum_{i=1}^l r_i, \quad C_l(\mathcal{D}) := \sum_{j=1}^l c_j.$$

As \mathcal{D} is usually a fixed brane diagram, we just denote $r_i(\mathcal{D})$, $c_j(\mathcal{D})$, $R_i(\mathcal{D})$ and $C_j(\mathcal{D})$ by r_i , c_j , R_i and C_j . The vectors $\mathbf{r} = (r_1, \dots, r_M)$ and $\mathbf{c} = (c_1, \dots, c_N)$ are called *margin vectors*. The following was proved in [RS24, Lemma 2.3]:

Lemma 2.9. *All $r_i(\mathcal{D})$ and $c_j(\mathcal{D})$ are non-negative and we have $R_M = C_N$.*

Let $\text{bct}(\mathcal{D})$ denote the set of all $M \times N$ matrices B with entries in $\{0, 1\}$ satisfying following row and column sum conditions:

- $\sum_{i=1}^M B_{i,j} = c_j$, for all $j \in \{1, \dots, N\}$,
- $\sum_{j=1}^N B_{i,j} = r_i$, for all $i \in \{1, \dots, M\}$.

The elements of $\text{bct}(\mathcal{D})$ are called *binary contingency tables of \mathcal{D}* .

The importance of binary contingency tables is the following bijection, see e.g. [Sho21, Proposition 2.2.8]

$$\text{Tie}(\mathcal{D}) \xleftrightarrow{1:1} \text{bct}(\mathcal{D}), \quad D \mapsto M(D), \quad D_B \leftarrow B. \quad (2.4)$$

The bijection is given as follows: if $D \in \text{Tie}(\mathcal{D})$ then the corresponding binary contingency table $M(D)$ is defined as

$$M(D)_{i,j} = \begin{cases} 1 & \text{if } V_i \triangleleft U_j \text{ and } (V_i, U_j) \in D, \\ 1 & \text{if } U_j \triangleleft V_i \text{ and } (U_j, V_i) \notin D, \\ 0 & \text{if } V_i \triangleleft U_j \text{ and } (V_i, U_j) \notin D, \\ 0 & \text{if } U_j \triangleleft V_i \text{ and } (U_j, V_i) \in D. \end{cases}$$

Conversely, if we are given $B \in \text{bct}(\mathcal{D})$, we obtain a tie diagram D_B as follows

$$D_B = D'_B \cup D''_B,$$

where

$$D'_B = \{(V_i, U_j) \mid V_i \triangleleft U_j, B_{i,j} = 1\}, \quad D''_B = \{(U_j, V_i) \mid U_j \triangleleft V_i, B_{i,j} = 0\}.$$

Next, we describe the separating line of a binary contingency table B which is a useful tool for illustrating the associated tie diagram D_B . For this, we draw the matrix B into a coordinate system where the entry $B_{i,j}$ is put into the square box with side length 1 and south-west corner at $(M - i, j - 1)$. Then, we define points p_0, \dots, p_{M+N} in this coordinate system via $p_0 = (0, 0)$ and

$$p_i = \begin{cases} p_{i-1} + (1, 0) & \text{if } X_i^- \text{ is blue,} \\ p_{i-1} + (0, 1) & \text{if } X_i^- \text{ is red.} \end{cases}$$

The *separating line* S_B of B is then obtained by connecting each p_i with p_{i+1} by a straight line. Using S_B we can easily illustrate D_B using the following rules:

- For each (i, j) such that $B_{i,j} = 1$ and the entry $B_{i,j}$ lies below S_D draw a dotted curve connecting V_i and U_j .
- For each (i, j) such that $B_{i,j} = 0$ and the entry $B_{i,j}$ lies above S_D draw a dotted curve connecting V_i and U_j .

Example 2.10. Let \mathcal{D} and D be as in Example 2.7. Then, the corresponding binary contingency table $M(D)$ with separating line is given as follows:

	U_1	U_2	U_3	U_4
V_1	1	0	0	1
V_2	0	1	1	0
V_3	0	0	1	1

2.7. Localization in equivariant cohomology. We now recall the localization principle in torus equivariant cohomology which is an astonishing feature of this cohomology theory that provides an interesting connection between local and global data. For more details on this subject see e.g. [Hsi75], [tD87], [AP93], [GKM98], [And12] and [AF23].

In the framework of bow varieties we are in the preferable situation of finitely many torus fixed points which are classified by tie diagrams respectively binary contingency tables as we discussed in the previous subsections.

We work with a fixed brane diagram \mathcal{D} and let $\mathbb{T} = \mathbb{A} \times \mathbb{C}_h^*$ be the torus from Subsection 2.3. Let $H_{\mathbb{T}}^*$ denote the \mathbb{T} -equivariant cohomology functor with coefficients in \mathbb{Q} . The \mathbb{T} -equivariant cohomology of a point $H_{\mathbb{T}}^*(\text{pt})$ is isomorphic to the polynomial algebra $\mathbb{Q}[t_1, \dots, t_N, h]$, where the parameters t_1, \dots, t_N correspond to \mathbb{A} and the parameter h corresponds to the factor \mathbb{C}_h^* .

Given a \mathbb{T} -fixed point $p \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$ we denote by $\iota_p^*: H_{\mathbb{T}}^*(\mathcal{C}(\mathcal{D})) \rightarrow H_{\mathbb{T}}^*(\{p\})$ the corresponding restriction morphism in equivariant cohomology. If $\gamma \in H_{\mathbb{T}}^*(\mathcal{C}(\mathcal{D}))$ then we call $\iota_p^*(\gamma)$ the *localization coefficient of γ at p* .

The equivariant localization theorem, see e.g. [AF23, Theorem 7.1.1], states that the restriction $\iota^*: H_{\mathbb{T}}^*(\mathcal{C}(\mathcal{D})) \rightarrow H_{\mathbb{T}}^*(\mathcal{C}(\mathcal{D})^{\mathbb{T}})$ induces an isomorphism

$$H_{\mathbb{T}}^*(\mathcal{C}(\mathcal{D}))_{\text{loc}} \xrightarrow{\sim} H_{\mathbb{T}}^*(\mathcal{C}(\mathcal{D})^{\mathbb{T}})_{\text{loc}},$$

where $H_{\mathbb{T}}^*(\mathcal{C}(\mathcal{D}))_{\text{loc}}$ resp. $H_{\mathbb{T}}^*(\mathcal{C}(\mathcal{D})^{\mathbb{T}})_{\text{loc}}$ is the localization at the multiplicative set generated by

$$\{a_1 t_1 + \dots + a_N t_N + b h \mid a_1, \dots, a_N, b \in \mathbb{Q}\} \subset H_{\mathbb{T}}^*(\text{pt}).$$

2.8. Tautological bundles. Bow varieties come with a family of *tautological bundles* which are \mathbb{T} -equivariant. As we discuss below, their first Chern classes generate the localized equivariant cohomology ring $H_{\mathbb{T}}^*(\mathcal{C}(\mathcal{D})^{\mathbb{T}})_{\text{loc}}$, see Corollary 2.13. Since the restrictions of tautological bundles fit well into the framework of tie diagrams, they form a preferred choice of generators for this cohomology ring.

Given a black line X in \mathcal{D} , the *tautological bundle of X* is defined as the geometric quotient

$$\xi_X := \mathbb{C}^{d_X} \times m^{-1}(0)^s / \mathcal{G}. \quad (2.5)$$

Here, \mathcal{G} and $m: \tilde{\mathcal{D}} \rightarrow \prod_{X' \in \text{ch}(\mathcal{D})} \text{End}(\mathbb{C}^{d_{X'}})$ are defined as in Subsection 2.2, \mathcal{G} acts diagonally on the product, where \mathcal{G} acts on \mathbb{C}^{d_X} via $(g_{X'})_{X'} \cdot v = g_X v$. Since the quotient morphism $m^{-1}(0)^s \rightarrow \mathcal{C}(\mathcal{D})$ is a principal \mathcal{G} -bundle (in the Zariski topology), we conclude that ξ_X is a vector bundle (in the Zariski topology) over $\mathcal{C}(\mathcal{D})$ of rank d_X . If $X = X_i$, we also denote ξ_X by ξ_i .

The torus \mathbb{T} acts on the first factor of $\mathbb{C}^{d_X} \times m^{-1}(0)^s$ which induces a \mathbb{T} -action on ξ_X giving ξ_X the structure of a \mathbb{T} -equivariant vector bundle over $\mathcal{C}(\mathcal{D})$.

The restrictions of tautological bundles to the \mathbb{T} -fixed points of $\mathcal{C}(\mathcal{D})$ can explicitly be computed via the following formula, see [RS24, Theorem 4.10]:

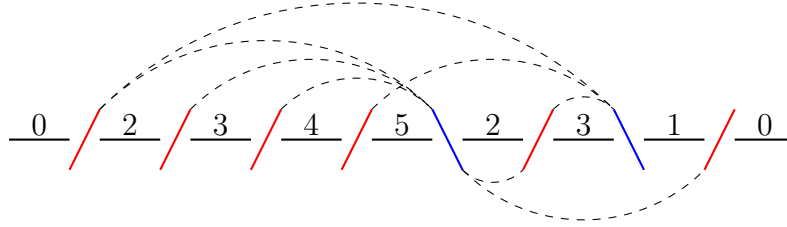
$$\iota_D^*(\xi_i) = \bigoplus_{U \in \text{b}(\mathcal{D})} \bigoplus_{k=0}^{d_{D,U,X_i}-1} h^{c_{D,U,j}-d_{D,U,U^-}+1+k} \mathbb{C}_U. \quad (2.6)$$

Here, \mathbb{C}_U is the one-dimensional \mathbb{A} -module given by the character $\mathbb{A} \rightarrow \mathbb{C}^*$ that projects to the factor with label U and $h^i \mathbb{C}_U$ is the \mathbb{T} -module by further letting \mathbb{C}_h^* act on \mathbb{C}_U via

$h.v = h^i v$ for $v \in \mathbb{C}_U$. The non-negative integer $d_{D,U,X}$ is defined for every black line X as $d_{D,U,X} = |\{V \in r(\mathcal{D}) \mid (V, U) \in D\}|$ for $X \triangleleft U$ and as $d_{D,U,X} = |\{V \in r(\mathcal{D}) \mid (U, V) \in D\}|$ for $U \triangleleft X$. The $c_{D,U,j}$ are given by $c_{D,U,j} := d_{D,U,U^+} - d_{D,U,X_j}$ for all j with $U \triangleleft X_j$. For j with $X_j \triangleleft U$, $c_{D,U,j}$ is defined recursively as

$$c_{D,U,j} := \begin{cases} c_{D,U,j} & \text{if } X_j^+ \text{ is blue,} \\ c_{D,U,j} & \text{if } X_j^+ \text{ is red and } d_{D,U,X_j} + 1 = d_{D,U,X_{j+1}}, \\ c_{D,U,j} - 1 & \text{if } X_j^+ \text{ is red and } d_{D,U,X_j} = d_{D,U,X_{j+1}}. \end{cases}$$

Example 2.11. Consider the brane diagram $0/2/3/4/5 \setminus 2/3 \setminus 1/0$ with tie diagram D :



For a blue line U and a black line X_j , the number d_{D,U,X_j} is the number of ties starting in U and covering X_j from above or below. For instance, there are three ties starting in U_1 and covering X_4 which gives $d_{D,U_1,X_4} = 3$. The other numbers d_{D,U,X_j} are recorded in the following table:

j	2	3	4	5	6	7	8
d_{D,U_1,X_j}	1	2	3	3	2	1	1
d_{D,U_2,X_j}	1	1	1	2	2	3	0

The resulting indices $c_{D,U,j}$ are then given as follows:

j	2	3	4	5	6	7	8
$c_{D,U_1,j}$	-1	-1	-1	0	0	1	1
$c_{D,U_2,j}$	-2	-1	0	0	0	0	0

Inserting in (2.6) then gives the restrictions of the tautological bundles to the \mathbb{T} -fixed point D . For example, since $d_{D,U_1,X_4} = 3$ and $c_{D,U_1,4} = -1$, the U_1 -contribution in (2.6) equals $h^{-3}\mathbb{C}_{U_1} \oplus h^{-2}\mathbb{C}_{U_1} \oplus h^{-1}\mathbb{C}_{U_1}$. Likewise, as $d_{D,U_1,X_4} = 1$ and $c_{D,U_1,4} = -1$, the U_2 contribution in (2.6) is $h^{-1}\mathbb{C}_{U_2}$. Consequently, $\iota_D^*(\xi_4) \cong h^{-3}\mathbb{C}_{U_1} \oplus h^{-2}\mathbb{C}_{U_1} \oplus h^{-1}\mathbb{C}_{U_1} \oplus h^{-2}\mathbb{C}_{U_2}$. The other restrictions $\iota_D^*(\xi_j)$ are given as follows:

j	$\iota_D^*(\xi_j)$
2	$h^{-3}\mathbb{C}_{U_1} \oplus h^{-4}\mathbb{C}_{U_2}$
3	$h^{-3}\mathbb{C}_{U_1} \oplus h^{-2}\mathbb{C}_{U_1} \oplus h^{-3}\mathbb{C}_{U_2}$
4	$h^{-3}\mathbb{C}_{U_1} \oplus h^{-2}\mathbb{C}_{U_1} \oplus h^{-1}\mathbb{C}_{U_1} \oplus h^{-2}\mathbb{C}_{U_2}$
5	$h^{-2}\mathbb{C}_{U_1} \oplus h^{-1}\mathbb{C}_{U_1} \oplus \mathbb{C}_{U_1} \oplus h^{-2}\mathbb{C}_{U_2} \oplus h^{-1}\mathbb{C}_{U_2}$
6	$h^{-2}\mathbb{C}_{U_1} \oplus h^{-1}\mathbb{C}_{U_1} \oplus h^{-2}\mathbb{C}_{U_2} \oplus h^{-1}\mathbb{C}_{U_2}$
7	$h^{-1}\mathbb{C}_{U_1} \oplus h^{-2}\mathbb{C}_{U_2} \oplus h^{-1}\mathbb{C}_{U_2} \oplus \mathbb{C}_{U_2}$
8	$h^{-1}\mathbb{C}_{U_1}$

Using (2.6), one can easily prove the following result:

Corollary 2.12. *Let $D, D' \in \text{Tie}(\mathcal{D})$. Then, $D = D'$ if and only if $\iota_D^*(\xi_X) = \iota_{D'}^*(\xi_X)$ for all $X \in \mathfrak{h}(\mathcal{D})$.*

Proof. If $\iota_D^*(\xi_X) = \iota_{D'}^*(\xi_X)$ for all $X \in \mathfrak{h}(\mathcal{D})$ then $d_{D,U,X} = d_{D',U,X}$ for all $U \in \mathfrak{b}(\mathcal{D})$ and $X \in \mathfrak{h}(\mathcal{D})$ by (2.6). This is equivalent to $D = D'$. \square

We conclude that the first Chern classes of the tautological bundles generate the localized equivariant cohomology $H_{\mathbb{T}}^*(\mathcal{C}(\mathcal{D}))_{\text{loc}}$:

Corollary 2.13. *We have that $c_1(\xi_1), \dots, c_1(\xi_{M+N+1})$ are $H_{\mathbb{T}}^*(\text{pt})_{\text{loc}}$ -algebra generators of $H_{\mathbb{T}}^*(\mathcal{C}(\mathcal{D}))_{\text{loc}}$.*

Proof. Let $A \subset H_{\mathbb{T}}^*(\mathcal{C}(\mathcal{D}))_{\text{loc}}$ be the $H_{\mathbb{T}}^*(\text{pt})_{\text{loc}}$ -algebra generated by $c_1(\xi_1), \dots, c_1(\xi_{M+N+1})$. By Corollary 2.12, we have for all $D, D' \in \text{Tie}(\mathcal{D})$ that $D = D'$ if and only if $\iota_D^*(c_1(\xi_X)) = \iota_{D'}^*(c_1(\xi_X))$ for all $X \in \mathfrak{h}(\mathcal{D})$. Thus, the Chinese remainder theorem implies that for all $D \in \text{Tie}(\mathcal{D})$ there exists an element $f_D \in A$ such that $f_D \in \bigcap_{D' \neq D} \ker(\iota_{D'}^*)$ and $f_D \equiv 1 \pmod{\ker(\iota_D^*)}$. Thus, the localization theorem implies $A = H_{\mathbb{T}}^*(\mathcal{C}(\mathcal{D}))_{\text{loc}}$. \square

Suppose now that \mathcal{D} is separated. As shown in e.g. [BR23, Proposition 3.4], the \mathbb{T} -equivariant vector bundles ξ_{U^-} are topologically trivial. The corresponding \mathbb{T} -characters given as follows:

Corollary 2.14. *Let \mathcal{D} be separated. Then, we have*

$$\xi_{U_j^-} = \bigoplus_{k=j}^N \bigoplus_{i=0}^{c_k-1} h^{-i} \mathbb{C}_{U_k}, \quad \text{for } U_j \in \mathfrak{b}(\mathcal{D}) \text{ and all } D \in \text{Tie}(\mathcal{D}).$$

Proof. Following the definition of the $d_{D,U,X}$ and $c_{D,U,X}$, the separatedness condition yields $d_{D,U_k,U_j} = 0$ if $k < j$ and $d_{D,U_k,U_j} = c_k$ if $k \geq j$. We also deduce $c_{D,U_k,M+j} = -c_k$ for $k \geq j$. Thus, Corollary 2.14 follows from (2.6). \square

2.9. Hanany–Witten transition. We consider now important classes of isomorphisms, so-called *Hanany–Witten isomorphisms*, corresponding to certain moves on brane diagrams: given brane diagrams \mathcal{D} and $\tilde{\mathcal{D}}$. Then, we say that $\tilde{\mathcal{D}}$ is obtained from \mathcal{D} via *Hanany–Witten transition* if $\tilde{\mathcal{D}}$ differs from \mathcal{D} by performing a local move of the form

where $d_1 + d_3 + 1 = d_2 + \tilde{d}_2$. Clearly, Hanany–Witten transitions reduce the separatedness degree (as defined in (2.1)) by 1. It is straightforward to show that from any (admissible) brane diagram we can obtain a separated brane diagram via a finite number of Hanany–Witten transitions, see e.g. [SW23, Proposition 4.12].

The following proposition, see [NT17, Proposition 7.1] and [RS24, Theorem 3.9], characterizes the isomorphism corresponding to a Hanany–Witten transition as well as the interplay of tautological bundles under this isomorphism:

Proposition 2.15. *Assume $\tilde{\mathcal{D}}$ is obtained from \mathcal{D} via a Hanany–Witten transition where the blue line U_j is exchanged with the red line V_i . Let X_k be the black line in \mathcal{D} with $X_k^- = U_j$ and $X_k^+ = V_i$. Then, there exists a ρ_j -equivariant isomorphism of symplectic varieties*

$$\Phi : \mathcal{C}(\mathcal{D}) \xrightarrow{\sim} \mathcal{C}(\tilde{\mathcal{D}}), \quad (2.7)$$

where ρ_j is the algebraic group automorphism

$$\rho_j : \mathbb{T} \rightarrow \mathbb{T}, \quad (t_1, \dots, t_N, h) \mapsto (t_1, \dots, t_{j-1}, ht_j, t_{j+1}, \dots, t_N, h).$$

Furthermore, the following holds:

- (i) We have \mathbb{T} -equivariant isomorphisms of vector bundles $\xi_l \cong \Phi^* \tilde{\xi}_l$ for $l \neq k$.
- (ii) There is a short exact sequence of \mathbb{T} -equivariant vector bundles

$$0 \rightarrow \xi_k \rightarrow \xi_{k-1} \oplus \xi_{k+1} \oplus h\mathcal{C}_{U_i} \rightarrow \Phi^* \tilde{\xi}_k \rightarrow 0. \quad (2.8)$$

Here, the $\tilde{\xi}_l$ denote the tautological bundles on $\mathcal{C}(\tilde{\mathcal{D}})$ and $\Phi^* \tilde{\xi}_l$ is the \mathbb{T} -equivariant pull-back of $\tilde{\xi}_l$ via Φ .

The fixed point matching under Hanany–Witten transition is described in [RS24, Section 4.7] as follows:

Proposition 2.16. *With the assumptions of Proposition 2.15, let $\phi : \mathcal{C}(\mathcal{D})^\mathbb{T} \xrightarrow{\sim} \mathcal{C}(\tilde{\mathcal{D}})^\mathbb{T}$ denote the bijection induced by the Hanany–Witten isomorphism Φ from (2.7). Then, we have $M(D) = M(\phi(D))$ for all $D \in \text{Tie}(\mathcal{D})$.*

The statement of tautological bundles from Proposition 2.15 directly gives the following relation of equivariant first Chern classes:

Corollary 2.17. *Under the assumptions of Proposition 2.15 let $\Phi^* : H_\mathbb{T}^*(\mathcal{C}(\tilde{\mathcal{D}})) \xrightarrow{\sim} H_\mathbb{T}^*(\mathcal{C}(\mathcal{D}))$ be the induced isomorphism of rings. Then, we have*

- (i) $\Phi^*(c_1(\tilde{\xi}_l)) = c_1(\xi_l)$ for $l \neq k$,
- (ii) $\Phi^*(c_1(\tilde{\xi}_k)) = c_1(\xi_{k+1}) + c_1(\xi_{k-1}) + h + t_j - c_1(\xi_k)$.

For a \mathbb{T} -equivariant vector bundle \mathcal{V} over $\mathcal{C}(\mathcal{D})$, we denote by $c_i(\mathcal{V}) \in H_\mathbb{T}^{2i}(\mathcal{C}(\mathcal{D}))$ its i -th \mathbb{T} -equivariant Chern class. By Corollary 2.17, the localization coefficients of first Chern classes of tautological bundles satisfy the following matching properties:

Corollary 2.18. *With the same notation as in Corollary 2.17 we have*

$$\varphi_j(\iota_{\phi(D)}^*(c_1(\tilde{\xi}_l))) = \iota_D^*(c_1(\xi_l)), \quad \text{for } l \neq k$$

and

$$\varphi_j(\iota_{\phi(D)}^*(c_1(\tilde{\xi}_k))) = \iota_D^*(c_1(\xi_{k+1}) + c_1(\xi_{k-1}) - c_1(\xi_k)) + h + t_j,$$

for all $D \in \text{Tie}(\mathcal{D})$, where $\varphi_j : \mathbb{Q}[t_1, \dots, t_N, h] \xrightarrow{\sim} \mathbb{Q}[t_1, \dots, t_N, h]$ is the $\mathbb{Q}[h]$ -algebra automorphism given by $t_j \mapsto t_j + h$ and $t_i \mapsto t_i$ for $i \neq j$.

3. COTANGENT BUNDLES OF PARTIAL FLAG VARIETIES AS BOW VARIETIES

For natural numbers $0 < d_1 < d_2 < \dots < d_m < n$ let $F(d_1, \dots, d_m; n)$ denote the partial flag variety parameterizing inclusions of \mathbb{C} -linear subspaces

$$\{0\} \subset E_1 \subset E_2 \subset \dots \subset E_m \subset \mathbb{C}^n$$

with $\dim(E_i) = d_i$ for $i = 1, \dots, m$.

It is well-known, see [Nak94, Theorem 7.3], that the cotangent bundle $T^*F(d_1, \dots, d_m; n)$ is isomorphic to the Nakajima quiver variety corresponding to the framed quiver

$$\begin{array}{ccccccc} \overset{1}{\bullet} & \longleftarrow & \overset{2}{\bullet} & \longleftarrow & \dots & \longleftarrow & \overset{m}{\bullet} \\ & & & & & & \uparrow \\ & & & & & & \square \end{array}$$

with dimension vector $(n - d_m, \dots, n - d_1)$, framing vector $(0, \dots, 0, n)$ and character

$$\theta: \prod_{i=1}^m \mathrm{GL}_{n-d_i}(\mathbb{C}) \longrightarrow \mathbb{C}^*, \quad (g_1, \dots, g_m) \mapsto \prod_{i=1}^m \det(g_i).$$

Thus, by [NT17, Theorem 2.15], we can realize $T^*F(d_1, \dots, d_m; n)$ as a bow variety. In this section, we will explicitly describe this realization. We also characterize the induced correspondence between the torus fixed point combinatorics of these varieties.

3.1. Realization via parabolic subgroups. Let $d_0 = 0$, $d_{m+1} = n$, $E_0 = 0$, $E_{m+1} = \mathbb{C}^n$ and $\delta_i = d_i - d_{i-1}$ for $i = 1, \dots, m+1$. Let $G = \mathrm{GL}_n(\mathbb{C})$ and $P \subset G$ be the parabolic subgroup of block matrices of the shape

$$\begin{pmatrix} P_{1,1} & P_{1,2} & \dots & P_{1,m+1} \\ & P_{2,2} & \dots & P_{2,m+1} \\ & & \ddots & \vdots \\ & & & P_{m+1,m+1} \end{pmatrix},$$

where each $P_{i,j}$ is a $\delta_i \times \delta_j$ matrix. It is well-known that the geometric quotient G/P exists and we have an isomorphism of varieties $G/P \xrightarrow{\sim} F(d_1, \dots, d_m; n)$ given by

$$[g] \mapsto (\{0\} \subset \langle g_1, \dots, g_{d_1} \rangle \subset \dots \subset \langle g_1, \dots, g_{d_m} \rangle \subset \mathbb{C}^n).$$

Here, g_i denotes the i -th column vector of g for $i = 1, \dots, n$.

Let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ be the Lie algebra of G and $\mathfrak{p} \subset \mathfrak{g}$ be the Lie-subalgebra corresponding to P . We denote by \mathfrak{p}^\perp the annihilator of \mathfrak{p} with respect to the trace pairing. That is, \mathfrak{p}^\perp is the Lie subalgebra of \mathfrak{g} consisting of block matrices of the form

$$\begin{pmatrix} 0 & P_{1,2} & P_{1,3} & \dots & P_{1,m+1} \\ & 0 & P_{2,3} & \dots & P_{2,m+1} \\ & & \ddots & \ddots & \vdots \\ & & & 0 & P_{m,m+1} \\ & & & & 0 \end{pmatrix}, \quad P_{i,j} \in \mathrm{Mat}_{\delta_i, \delta_j}(\mathbb{C}).$$

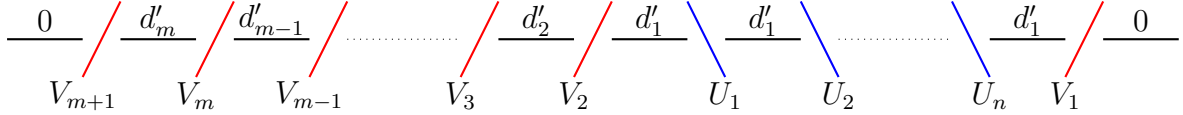
Here, again $P_{i,j}$ is a $\delta_i \times \delta_j$ matrix. The parabolic subgroup P acts algebraically on \mathfrak{p}^\perp via conjugation. It is well-known that the cotangent bundle $T^*F(d_1, \dots, d_m; n)$ is isomorphic

as algebraic variety to the geometric quotient $(G \times \mathfrak{p}^\perp)/P$, see e.g. [CG97, Lemma 1.4.9]. Hence, the points of $T^*F(d_1, \dots, d_m; n)$ can be identified with pairs (\mathcal{F}, f) where

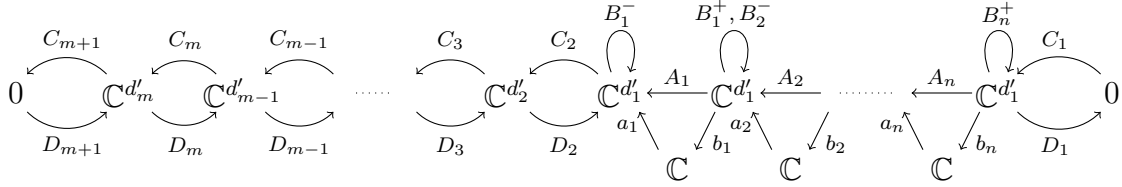
$$\mathcal{F} = (\{0\} \subset E_1 \subset \dots \subset E_m \subset \mathbb{C}^n)$$

is a point in $F(d_1, \dots, d_m; n)$ and $f \in \text{End}(\mathbb{C}^n)$ such that $f(E_i) \subset E_{i-1}$ for $i = 1, \dots, m+1$.

3.2. Bow variety realization. Let $\tilde{\mathcal{D}}(d_1, \dots, d_m; n)$ be the brane diagram:



where $d'_i = n - d_i$ for $i = 1, \dots, m$. We denote elements of $\mathcal{C}(\tilde{\mathcal{D}}(d_1, \dots, d_m; n))$ according to the diagram



Given $x = [(A_i, B_i^+, B_i^-, a_i, b_i)_i; (C_j, D_j)_j] \in \mathcal{C}(\tilde{\mathcal{D}}(d_1, \dots, d_m; n))$. By [Tak16, Lemma 2.18], the linear operators A_1, \dots, A_n are isomorphisms. Hence, we can define the operators

$$a: \mathbb{C}^n \longrightarrow \mathbb{C}^{d'_1}, \quad b: \mathbb{C}^{d'_1} \longrightarrow \mathbb{C}^n$$

via the matrices

$$a = (a_1 \quad A_1 a_2 \quad \dots \quad A_1 \cdots A_{n-1} a_n), \quad b = \begin{pmatrix} b_1 A_1^{-1} \\ b_2 A_2^{-1} A_1^{-1} \\ \vdots \\ b_{n-1} A_{n-1}^{-1} \cdots A_1^{-1} \\ b_n A_n^{-1} \cdots A_1^{-1} \end{pmatrix}.$$

The stability criterion [NT17, Proposition 2.8] implies that

$$\mathcal{F}_x := (\{0\} \subset \ker(a) \subset \ker(C_1 a) \subset \dots \subset \ker(C_1 \dots C_m a) \subset \mathbb{C}^n)$$

defines a point on $F(d_1, \dots, d_m; n)$ which is independent of choice of representative for x . By construction, the linear operator $f_x \in \text{End}(\mathbb{C}^n)$, $f_x := ba$ is also independent of the choice of an representative for x . The isomorphism

$$\Psi: \mathcal{C}(\tilde{\mathcal{D}}(d_1, \dots, d_m; n)) \xrightarrow{\sim} T^*F(d_1, \dots, d_m; n) \quad (3.1)$$

from [NT17, Theorem 2.15] is then given by

$$\Psi(x) = (\mathcal{F}_x, f_x), \quad \text{for all } x \in \mathcal{C}(\tilde{\mathcal{D}}(d_1, \dots, d_m; n)).$$

3.3. Torus action and tautological bundles. The $\mathbb{T} = (\mathbb{A} \times \mathbb{C}_h^*)$ -action on $\mathcal{C}(\mathcal{D})$ from Subsection 2.3 induces the following \mathbb{T} -action on $T^*F(d_1, \dots, d_m; n)$:

$$t.(\mathcal{F}, f) = (d(t)(\mathcal{F}), d(t)fd(t)^{-1}), \quad h.(\mathcal{F}, f) = (\mathcal{F}, hf),$$

where $t = (t_1, \dots, t_n) \in \mathbb{A}$, $(\mathcal{F}, f) \in T^*F(d_1, \dots, d_m; n)$ and $d(t)$ is the diagonal operator such that $d(t)(e_i) = t_i e_i$ for $i = 1, \dots, n$, where e_1, \dots, e_n denote the standard basis vectors of \mathbb{C}^n .

For $i \in \{1, \dots, m\}$, let

$$\mathcal{S}_i = \{(\{0\} \subset E_1 \subset E_2 \subset \dots \subset E_m \subset \mathbb{C}^n), v) \mid v \in E_i\} \subset F(d_1, \dots, d_m; n) \times \mathbb{C}^n$$

be the corresponding tautological bundle and $\mathcal{Q}_i = (F(d_1, \dots, d_m; n) \times \mathbb{C}^n)/\mathcal{S}_i$ the quotient bundle. By abuse of language, we also denote the pullbacks of \mathcal{S}_i and \mathcal{Q}_i to $T^*F(d_1, \dots, d_m; n)$ by \mathcal{S}_i and \mathcal{Q}_i . Both, \mathcal{S}_i and \mathcal{Q}_i are \mathbb{T} -equivariant vector bundles over $T^*F(d_1, \dots, d_m; n)$, where the \mathbb{T} -action is induced by the \mathbb{T} -action on $T^*F(d_1, \dots, d_m; n) \times \mathbb{C}^n$ where \mathbb{A} acts on \mathbb{C}^n via the standard action and \mathbb{C}_h^* acts trivially on \mathbb{C}^n .

From the construction of Ψ , it follows that $\Psi^*\mathcal{Q}_i = h^{i-1}\xi_{m-i+1}$, so up to a scaling factor the tautological bundles on the bow variety $\mathcal{C}(\tilde{\mathcal{D}}(d_1, \dots, d_m; n))$ correspond to the quotient bundles on $T^*F(d_1, \dots, d_m; n)$.

3.4. Fixed point matching. In this subsection, we describe the bijection of \mathbb{T} -fixed points

$$T^*F(d_1, \dots, d_m; n)^{\mathbb{T}} \xleftarrow{1:1} \mathcal{C}(\tilde{\mathcal{D}}(d_1, \dots, d_m; n))^{\mathbb{T}}. \quad (3.2)$$

that is induced by the isomorphism Ψ from (3.1).

Let S_n the symmetric group on n letters. We usually denote permutations $w \in S_n$ in one line notation $w = w(1)w(2) \dots w(n)$.

For a permutation $w \in S_n$, we define the flag

$$\mathcal{F}_w := (\{0\} \subset \langle e_{w(1)}, \dots, e_{w(d_1)} \rangle \subset \dots \subset \langle e_{w(1)}, \dots, e_{w(d_m)} \rangle \subset \mathbb{C}^n).$$

It is well-known that we have a bijection

$$S_n/S_{\delta} \xrightarrow{\sim} (T^*F(d_1, \dots, d_m; n))^{\mathbb{T}}, \quad wS_{\delta} \mapsto (\mathcal{F}_w, 0).$$

where $S_{\delta} = S_{\delta_1} \times \dots \times S_{\delta_{m+1}} \subset S_n$ is the Young subgroup corresponding to $\delta = (\delta_1, \dots, \delta_{m+1})$.

Following the construction of Ψ reveals that the preimage $\Psi^{-1}(\mathcal{F}_w, 0)$ corresponds to the diagram is represented by

$$[(A_{w,i}, B_{w,i}^+, B_{w,i}^-, a_{w,i}, b_{w,i})_i; (C_{w,j}, D_{w,j})_j],$$

where $A_{w,i} = \text{id}_{\mathbb{C}^{d'_1}}$ for all i and

$$\begin{aligned} a_{w,i}: \mathbb{C} &\rightarrow \mathbb{C}^{d'_1}, \quad 1 \mapsto 0, \quad \text{for } i \in \{w(1), \dots, w(d_1)\}, \\ a_{w,i}: \mathbb{C} &\rightarrow \mathbb{C}^{d'_1}, \quad 1 \mapsto e_{w^{-1}(i)-d_1}, \quad \text{for } i \in \{w(d_1+1), \dots, w(n)\}, \\ C_{w,j}: \mathbb{C}^{d'_{j-1}} &\rightarrow \mathbb{C}^{d'_j} \quad C_{w,j}(e_i) = \begin{cases} 0 & \text{if } i = 1, \dots, d_j, \\ e_{i-d'_j} & \text{if } i = d_j+1, \dots, d'_j, \end{cases} \quad \text{for } j = 2, \dots, m+1. \end{aligned}$$

The remaining operators vanish.

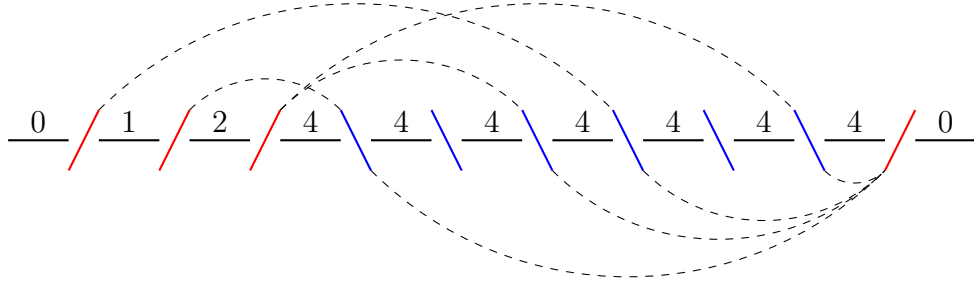
Following the explicit construction of \mathbb{T} -fixed points of bow varieties from [RS24, Section 4] gives $\Psi^{-1}(\mathcal{F}_w, 0) = x_{\tilde{D}_w}$, where $\tilde{D}_w = \tilde{D}'_w \cup \tilde{D}''_w$. Here, \tilde{D}'_w is the set of all pairs (V_i, U_j) with

$i \in \{2, 3, \dots, m\}, j \in \{1, \dots, n\}$ and there exists $l \in \{d_{i-1} + 1, \dots, d_i\}$ such that $w(l) = j$. The set \tilde{D}_w'' is defined as the set of all pairs (U_j, V_1) with $j \in \{1, \dots, n\}$ and there exists $l \in \{d_1 + 1, \dots, n\}$ such that $w(l) = j$.

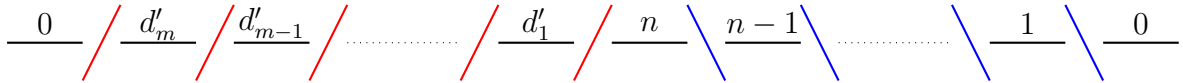
A straight-forward check shows that \tilde{D}_w only depends on the coset wS_δ . Hence, we denote \tilde{D}_w also by \tilde{D}_{wS_δ} . Thus, (3.2) corresponds to the following combinatorial bijection:

$$S_n/S_\delta \xrightarrow{\sim} \text{Tie}(\tilde{\mathcal{D}}(d_1, \dots, d_m; n)), \quad wS_\delta \mapsto D_{wS_\delta}. \quad (3.3)$$

Example 3.1. Let $m = 3, d_1 = 2, d_2 = 4, d_3 = 5$ and $n = 6$. Then, the brane diagram $\tilde{\mathcal{D}}(2, 4, 5; 6)$ equals $0/1/2/4 \backslash 4 \backslash 4 \backslash 4 \backslash 4 \backslash 4/0$. Let $w \in S_6$ be the permutation $w = 253614$. To construct \tilde{D}_w , note that since $d_1 = 2$, we have that there are no ties in \tilde{D}_w which are connected to U_2 and U_5 . As $d_2 = 4$, the blue lines U_3, U_6 are both connected to V_1 and V_2 . Likewise, since $d_3 = 5$, the blue line U_1 is connected to V_1 and V_3 . Finally, $n = 6$ implies that there are ties between U_4 and V_1, V_4 . Hence, \tilde{D}_w is illustrated as follows:



3.5. Transition to separated brane diagram. In the later course of this article, we will mostly work with bow varieties corresponding to separated brane diagrams. We define the brane diagram $\mathcal{D}(d_1, \dots, d_m; n)$ as follows:



Note that $\mathcal{D}(d_1, \dots, d_m; n)$ is obtained from $\tilde{\mathcal{D}}(d_1, \dots, d_m; n)$ via Hanany–Witten transitions by moving V_1 to the left of U_1, \dots, U_n . Let

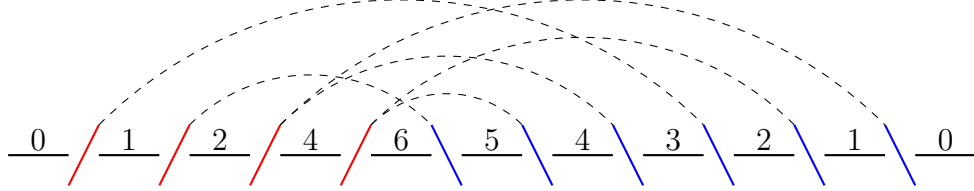
$$\Phi: \mathcal{C}(\mathcal{D}(d_1, \dots, d_m; n)) \xrightarrow{\sim} \mathcal{C}(\tilde{\mathcal{D}}(d_1, \dots, d_m; n)) \quad (3.4)$$

be the corresponding Hanany–Witten isomorphism. Then, by Proposition 2.16, the \mathbb{T} -fixed point matching under Φ can be characterized as follows: for $w \in S_n$, let D_w be the tie diagram attached to $\mathcal{D}(d_1, \dots, d_m; n)$ consisting of all pairs (V_i, U_j) with $i \in 1, \dots, m+1$, $j \in \{1, \dots, n\}$ and there exists $l \in \{d_{i-1} + 1, \dots, d_i\}$ such that $w(l) = j$. Again, the D_w only depends on the coset wS_δ . Employing Proposition 2.16 gives that for all $w \in S_n$, Φ maps $x_{\tilde{D}_{wS_\delta}}$ to $x_{D_{wS_\delta}}$. Thus, by (3.3), we have a bijection

$$S_n/S_\delta \xrightarrow{\sim} \text{Tie}(\mathcal{D}(d_1, \dots, d_m; n)), \quad wS_\delta \mapsto D_{wS_\delta}.$$

Example 3.2. As in Example 3.1, we choose $m = 3, d_1 = 2, d_2 = 4, d_3 = 5$ and $n = 6$. Thus, $\mathcal{D}(2, 4, 5; 6)$ is given by $0/1/2/4/6 \backslash 5 \backslash 4 \backslash 3 \backslash 2 \backslash 1/0$. Again, choose $w = 253614$. For the construction of D_w , note that since $d_1 = 2$, the blue lines U_2 and U_5 are connected in D_w

to V_1 . As $d_2 = 4$, the blue lines U_3 and U_6 are connected to V_2 . Likewise, $d_3 = 5$ gives that there is a tie between U_1 and V_3 . Finally, $n = 6$ implies that U_4 is connected to V_4 . Therefore, we can illustrate D_w as follows:



4. STABLE ENVELOPES

Stable envelopes are families of equivariant cohomology classes introduced by Maulik and Okounkov in [MO19]. They exist for a large class of symplectic varieties with torus action including Nakajima quiver varieties and more generally bow varieties. Their definition involves stability conditions which are similar to the stability conditions of equivariant Schubert classes, see e.g. [KT03], [GKS20].

In this section, we recall the definition of stable envelopes in the framework of bow varieties and some of their properties.

4.1. Attracting cells. The theory of stable envelopes is based on the theory of attracting cells. We briefly recall important features of this theory.

Notation. Let V be a finite dimensional representation of \mathbb{C}^* and

$$V_a = \{v \in V \mid t.v = t^a v \text{ for all } t \in \mathbb{C}^*\}, \quad \text{for } a \in \mathbb{Z}.$$

We set $V^+ := \bigoplus_{a \geq 1} V_a$ and $V^- := \bigoplus_{a \leq -1} V_a$.

As in Subsection 2.5, let σ be a generic cocharacter of \mathbb{A} . The *attracting cell* (with respect to σ) is defined as

$$\text{Attr}_\sigma(p) = \{x \in \mathcal{C}(\mathcal{D}) \mid \lim_{t \rightarrow 0} \sigma(t).x = p\}.$$

Using the classical Białnicki-Birula theorem, one can show that $\text{Attr}_\sigma(p)$ is an affine and locally closed \mathbb{T} -invariant subvariety of $\mathcal{C}(\mathcal{D})$ which is \mathbb{T} -equivariantly isomorphic to $T_p \mathcal{C}(\mathcal{D})_\sigma^+$, see [MO19, Lemma 3.2.4] and also the exposition in [SW23, Proposition 5.1]. Here, $T_p \mathcal{C}(\mathcal{D})_\sigma$ is the \mathbb{C}^* -representation obtained from the \mathbb{A} -representation $T_p \mathcal{C}(\mathcal{D})$ via σ .

Attracting cells admit the following independence property: let Λ be the cocharacter lattice of \mathbb{A} and set $\Lambda_{\mathbb{R}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. We have the usual root hyperplanes of $\mathfrak{gl}_N(\mathbb{C})$ in $\Lambda_{\mathbb{R}}$:

$$H_{i,j} := \{(t_1, \dots, t_N) \mid t_i = t_j\} \subset \Lambda_{\mathbb{R}}, \quad \text{for } 1 \leq i, j \leq N \text{ with } i \neq j.$$

The *chambers* of $\mathfrak{gl}_N(\mathbb{C})$ are the connected components of

$$\Lambda_{\mathbb{R}} \setminus \left(\bigcup_{\substack{1 \leq i, j \leq N \\ i \neq j}} H_{i,j} \right)$$

We have the *dominant chamber* and the *antidominant chamber*

$$\mathfrak{C}_+ = \{(t_1, \dots, t_N) \mid t_1 > t_2 > \dots > t_N\}, \quad \mathfrak{C}_- = \{(t_1, \dots, t_N) \mid t_1 < t_2 < \dots < t_N\}.$$

As usual, reflecting along the root hyperplanes gives an action of the symmetric group S_N on $\Lambda_{\mathbb{R}}$. It coincides with the action which permutes the coordinates and provides a well-known bijection from Lie theory

$$\{\text{Chambers}\} \xleftarrow{1:1} S_N,$$

where we assign to a permutation $w \in S_N$ the chamber $w \cdot \mathfrak{C}_+$.

Attracting cells are constant on chambers, that is if σ, τ lie in the same chamber \mathfrak{C} , we have $\text{Attr}_{\sigma}(p) = \text{Attr}_{\tau}(p)$, see e.g. [SW23, Proposition 5.4]. Hence, we also just write $\text{Attr}_{\mathfrak{C}}(p)$. In addition, $T_p \mathcal{C}(\mathcal{D})_{\sigma}^{\pm}$ also just depends on \mathfrak{C} . Therefore, we also just write $T_p \mathcal{C}(\mathcal{D})_{\mathfrak{C}}^{\pm}$.

There is a partial order $\preceq = \preceq_{\mathfrak{C}}$ on $\mathcal{C}(\mathcal{D})^{\mathbb{T}}$ defined as the transitive closure of the relation

$$q \in \overline{\text{Attr}_{\mathfrak{C}}(p)} \Rightarrow q \preceq p,$$

where $\overline{\text{Attr}_{\mathfrak{C}}(p)}$ denotes the Zariski closure of $\text{Attr}_{\mathfrak{C}}(p)$ in $\mathcal{C}(\mathcal{D})$. For any $p \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$, we set

$$\text{Attr}_{\mathfrak{C}}^f(p) := \bigsqcup_{q \preceq p} \text{Attr}_{\mathfrak{C}}(q).$$

As in the case of Schubert varieties, one can show that $\text{Attr}_{\mathfrak{C}}^f(p)$ is a closed subvariety of $\mathcal{C}(\mathcal{D})$ which is called the *full attracting cell* of p , see [MO19, Lemma 3.2.7] or the exposition in [SW23, Proposition 5.8].

The *opposite chamber* of \mathfrak{C} is defined as

$$\mathfrak{C}^{\text{op}} := \{a \in \mathfrak{a}_{\mathbb{R}} \mid -a \in \mathfrak{C}\}.$$

It is a general fact that $\preceq_{\mathfrak{C}^{\text{op}}}$ is the opposite order of $\preceq_{\mathfrak{C}}$, see e.g. [SW23, Proposition 5.10].

4.2. Definition of stable envelopes. Let $d = \dim(\mathcal{C}(\mathcal{D}))$ be the dimension of $\mathcal{C}(\mathcal{D})$ as complex variety.

Stable envelopes are maps

$$\mathcal{C}(\mathcal{D})^{\mathbb{T}} \xrightarrow{\text{Stab}_{\mathfrak{C}}} H_{\mathbb{T}}^d(\mathcal{C}(\mathcal{D})),$$

depending on a choice of chamber \mathfrak{C} which are uniquely characterized by the following stability conditions, see [MO19, Theorem 3.3.4]:

Theorem 4.1. *There exist a unique family $(\text{Stab}_{\mathfrak{C}}(p))_{p \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}}$ in $H_{\mathbb{T}}^d(\mathcal{C}(\mathcal{D}))$ satisfying the following conditions:*

- (Stab1) *We have $\iota_p^*(\text{Stab}_{\mathfrak{C}}(p)) = e_{\mathbb{T}}(T_p \mathcal{C}(\mathcal{D})_{\mathfrak{C}}^-)$, for all $p \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$.*
- (Stab2) *We have that $\text{Stab}_{\mathfrak{C}}(p)$ is supported on $\text{Attr}_{\mathfrak{C}}^f(p)$, for all $p \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$.*
- (Stab3) *Given $p, q \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}$ with $q \prec p$. Then, $\iota_q^*(\text{Stab}_{\mathfrak{C}}(p))$ is divisible by h .*

Here, $e_{\mathbb{T}}$ denotes the \mathbb{T} -equivariant Euler class.

The condition (Stab1) is called *normalization condition*, (Stab2) is called *support condition* and (Stab3) is called *smallness condition*.

By the normalization and support condition, we obtain that $(\text{Stab}_{\mathfrak{C}}(p))_{p \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}}$ is a basis of the localized equivariant cohomology ring $H_{\mathbb{T}}^*(\mathcal{C}(\mathcal{D}))_{\text{loc}}$. The basis $(\text{Stab}_{\mathfrak{C}}(p))_{p \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}}$ is called the *stable basis (corresponding to \mathfrak{C})*. We refer to the equivariant cohomology classes $\text{Stab}_{\mathfrak{C}}(p) \in H_{\mathbb{T}}^d(\mathcal{C}(\mathcal{D}))$ as *stable basis elements*.

Remark 4.2. In [MO19] the definition of stable envelopes slightly differs from our definition given in Theorem 4.1. Their version of the normalization condition (Stab1) involves a certain choice of signs depending on a choice of polarization bundle of the bow variety. It was shown in [Sho21, Section 4] that polarization bundles always exist for bow varieties. For simplicity, we chose all signs in the normalization condition (Stab1) to be +1.

In the next subsections, we state convenient properties of stable bases.

4.3. Orthogonality. As shown in [MO19, Theorem 4.4.1] (see also [SW23, Theorem 8.1]), stable bases satisfy the following useful orthogonality property which is analogous to the orthogonality properties of Schubert bases:

Theorem 4.3 (Orthogonality). *For all $p, q \in \mathcal{C}(\mathcal{D})^\mathbb{T}$ we have*

$$(\text{Stab}_{\mathfrak{C}}(p), \text{Stab}_{\mathfrak{C}^{\text{op}}}(q))_{\text{virt}} = \begin{cases} 1 & \text{if } p = q, \\ 0 & \text{if } p \neq q, \end{cases}$$

where

$$(\cdot, \cdot)_{\text{virt}} : H_{\mathbb{T}}^*(\mathcal{C}(\mathcal{D})) \times H_{\mathbb{T}}^*(\mathcal{C}(\mathcal{D})) \rightarrow S^{-1}H_{\mathbb{T}}^*(\text{pt}), \quad (\alpha, \beta)_{\text{virt}} = \sum_{p \in \mathcal{C}(\mathcal{D})^\mathbb{T}} \frac{\iota_p^*(\alpha \cup \beta)}{e_{\mathbb{T}}(T_p \mathcal{C}(\mathcal{D}))}$$

is the virtual intersection pairing with

$$S = \{t_i - t_j + mh \mid i, j \in \{1, \dots, N\}, m \in \mathbb{Z}\}. \quad (4.1)$$

Remark 4.4. It is well-known that the tangent weights of fixed points of $\mathcal{C}(\mathcal{D})$ are contained in S , see e.g. [SW23, Corollary 4.15]. Thus, the virtual intersection form takes indeed values in $S^{-1}H_{\mathbb{T}}^*(\text{pt})$.

The following practical property of the stable basis on matrix coefficients of multiplication operators of equivariant cohomology classes with respect to the stable basis can be found in [SW23, Proposition 8.2]:

Proposition 4.5 (Polynomiality). *For all $\gamma \in H_{\mathbb{T}}^*(\mathcal{C}(\mathcal{D}))$ and all $p, q \in \mathcal{C}(\mathcal{D})^\mathbb{T}$ we have*

$$(\gamma \cup \text{Stab}_{\mathfrak{C}}(p), \text{Stab}_{\mathfrak{C}^{\text{op}}}(q))_{\text{virt}} \in H_{\mathbb{T}}^*(\text{pt}).$$

4.4. Matching under Hanany–Witten transition. Suppose $\tilde{\mathcal{D}}$ is obtained from \mathcal{D} via Hanany–Witten transition and let $\Phi: \mathcal{C}(\mathcal{D}) \xrightarrow{\sim} \mathcal{C}(\tilde{\mathcal{D}})$ be the corresponding Hanany–Witten isomorphism and $\phi: \text{Tie}(\mathcal{D}) \xrightarrow{\sim} \text{Tie}(\tilde{\mathcal{D}})$ be the induced bijection.

Stable bases are compatible with Hanany–Witten transition:

Proposition 4.6. *Let $\Phi^*: H_{\mathbb{T}}^*(\mathcal{C}(\tilde{\mathcal{D}})) \xrightarrow{\sim} H_{\mathbb{T}}^*(\mathcal{C}(\mathcal{D}))$ be the induced isomorphism. Then, we have for all $D \in \text{Tie}(\mathcal{D})$*

$$\Phi^*(\text{Stab}_{\mathfrak{C}}(\phi(D))) = \text{Stab}_{\mathfrak{C}}(D).$$

Proof. The smallness condition is immediate from $\Phi^*(h) = h$. As Φ is \mathbb{A} -equivariant, we have

$$\Phi^{-1}(\text{Attr}_{\mathfrak{C}}(\phi(D))) = \text{Attr}_{\mathfrak{C}}(D), \quad \text{for all } D \in \text{Tie}(\mathcal{D}) \quad (4.2)$$

which implies the support condition. In addition, (4.2) gives

$$e_{\mathbb{T}}(T_D \mathcal{C}(\mathcal{D})_{\mathfrak{C}}^-) = \iota_D^*(\overline{\text{Attr}_{\mathfrak{C}}}(D)) = \iota_D^*(\Phi^*(\overline{\text{Attr}_{\mathfrak{C}}}(\phi(D))))$$

which proves the normalization condition. \square

5. STABLE BASES IN THE SEPARATED CASE

In this section, we recall some convenient properties of stable basis elements of bow varieties corresponding to separated brane diagrams from [BR23]. In this reference, the authors work in the framework of elliptic cohomology. As explained in e.g. [Weh24], the same results also hold in torus equivariant cohomology.

Assumption. For this section, we assume that \mathcal{D} is a fixed separated brane diagram.

5.1. Forgetting chargeless lines. We call a colored line Y in \mathcal{D} *chargeless* if $d_{Y+} = d_{Y-}$. That is, the horizontal lines to the left and to the right of Y are labeled by the same number. If Y is not chargeless then Y is called *essential*. If all colored lines of \mathcal{D} are essential we also call \mathcal{D} *essential*.

Let $\text{ess}(\mathcal{D})$ be the brane diagram obtained from \mathcal{D} by forgetting all chargeless lines. That is for all chargeless lines in \mathcal{D} we perform the local moves:

$$\text{---} \frac{d}{\text{red}} \text{---} \quad \rightsquigarrow \quad \text{---} \frac{d}{\text{blue}} \text{---} \quad \text{respectively} \quad \text{---} \frac{d}{\text{blue}} \text{---} \quad \rightsquigarrow \quad \text{---} \frac{d}{\text{red}} \text{---}$$

For instance if $\mathcal{D} = 0/2/2/4/5/5/4/2/0$ then $\text{ess}(\mathcal{D}) = 0/2/4/5/4/2/0$.

Note that in a tie diagram $D \in \text{Tie}(\mathcal{D})$ there is no tie which is connected to a chargeless line. Thus, forgetting the chargeless lines gives a bijection

$$f: \text{Tie}(\mathcal{D}) \xrightarrow{\sim} \text{Tie}(\text{ess}(\mathcal{D})).$$

We also view $\mathcal{C}(\text{ess}(\mathcal{D}))$ as \mathbb{T} -variety, where the components of \mathbb{A} corresponding to chargeless lines act trivially on $\mathcal{C}(\text{ess}(\mathcal{D}))$.

The localization coefficients of stable basis elements of $\mathcal{C}(\mathcal{D})$ and $\mathcal{C}(\text{ess}(\mathcal{D}))$ are closely connected, see [BR23, Section 5.10] or [Weh24, Proposition 8.44]:

Proposition 5.1. *The following holds:*

- (i) *There is a \mathbb{T} -equivariant closed immersion $\iota: \mathcal{C}(\text{ess}(\mathcal{D})) \hookrightarrow \mathcal{C}(\mathcal{D})$.*
- (ii) *The \mathbb{T} -equivariant cohomology class of the normal bundle of ι equals $[N_\iota]$, where*

$$N_\iota = \bigoplus_{U \in \mathbf{b}'(\mathcal{D})} \left(\text{Hom}(\mathbb{C}_U, \xi_{U-}) \oplus h \text{Hom}(\xi_{U+}, \mathbb{C}_U) \right)$$

and $\mathbf{b}'(\mathcal{D})$ denotes the set of chargeless blue lines in \mathcal{D} .

- (iii) *We have*

$$\iota_{D'}^* \text{Stab}_{\mathfrak{C}}(D) = e_{\mathbb{T}}(N_{\iota, \mathfrak{C}}^-) \iota_{f(D')}^* (\text{Stab}_{\mathfrak{C}}(f(D))),$$

for all $D, D' \in \text{Tie}(\mathcal{D}')$, where $N_{\iota, \mathfrak{C}}^-$ is the negative part of N_ι with respect to \mathfrak{C} .

It follows from Corollary 2.14 that N_ι is trivial with character

$$N_\iota = \left(\bigoplus_{U_j \in \mathbf{b}'(\mathcal{D})} \bigoplus_{k=j+1}^N \bigoplus_{l=0}^{c_k-1} h^{-l} \mathbb{C}_{U_k} \otimes \mathbb{C}_{U_j}^\vee \right) \oplus \left(\bigoplus_{U_j \in \mathbf{b}'(\mathcal{D})} \bigoplus_{k=j+1}^N \bigoplus_{l=0}^{c_k} h^{l+1} \mathbb{C}_{U_j} \otimes \mathbb{C}_{U_k}^\vee \right), \quad (5.1)$$

where the left summand corresponds to $\text{Hom}(\mathbb{C}_U, \xi_{U-})$ whereas the right summand corresponds to $h \text{Hom}(\xi_{U+}, \mathbb{C}_U)$. In the above formula, $\mathbb{C}_{U_k}^\vee$ denotes the dual bundle of \mathbb{C}_{U_k} .

If we choose $\mathfrak{C} = \mathfrak{C}_-$ to be the antidominant chamber then (5.1) implies

$$N_{\iota, \mathfrak{C}_-}^+ = \bigoplus_{U \in \mathfrak{b}'(\mathcal{D})} \text{Hom}(\mathbb{C}_U, \xi_{U^-}), \quad N_{\iota, \mathfrak{C}_-}^- = \bigoplus_{U \in \mathfrak{b}'(\mathcal{D})} h \text{Hom}(\xi_{U^+}, \mathbb{C}_U).$$

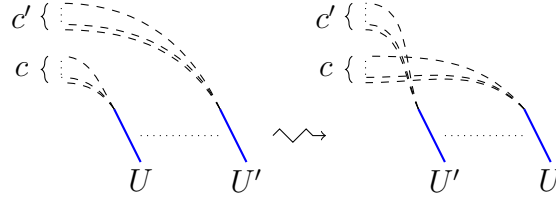
Thus, we have

$$e_{\mathbb{T}}(N_{\iota, \mathfrak{C}_-}^+) = \prod_{U_j \in \mathfrak{b}'(\mathcal{D})} \prod_{k=j+1}^N \prod_{l=0}^{c_k-1} (t_k - t_j - lh), \quad e_{\mathbb{T}}(N_{\iota, \mathfrak{C}_-}^-) = \prod_{U_j \in \mathfrak{b}'(\mathcal{D})} \prod_{k=j+1}^N \prod_{l=0}^{c_k-1} (t_j - t_k + (l+1)h)$$

which determines the normalization factor in Proposition 5.1 for the antidominant chamber.

5.2. Compatibility with the symmetric group action. In this subsection, we assume that \mathcal{D} is essential as defined in the previous subsection.

Given a tie diagram $D \in \text{Tie}(\mathcal{D})$ and two blue lines $U, U' \in \mathfrak{b}(\mathcal{D})$. Then, swapping the blue lines U, U' with their connected ties gives a new tie diagram over a brane diagram that possibly differs from \mathcal{D} :



This gives S_N -actions on the sets

$$B_N := \{\text{Separated brane diagrams } \mathcal{D} \mid |\mathfrak{b}(U)| = N\} \quad \text{and} \quad \bigcup_{\mathcal{D} \in B_N} \text{Tie}(\mathcal{D}).$$

Given a permutation $w \in S_N$ then $w.\mathcal{D}$ is the separated brane diagram with M red lines, N blue lines and the numbers on the horizontal lines are given as

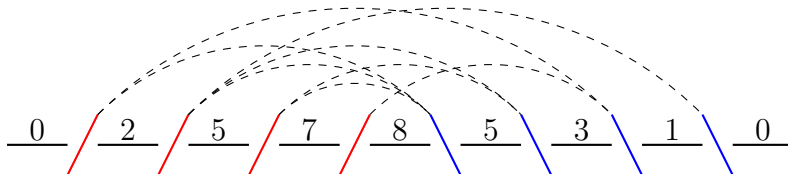
$$d_{X_i}(w.\mathcal{D}) = d_{X_i}(\mathcal{D}), \quad i = 1, \dots, M+1, \quad d_{X_{M+j}}(w.\mathcal{D}) = \sum_{l=j}^N c_{w^{-1}(l)}(\mathcal{D}), \quad j = 1, \dots, N+1.$$

By construction, $\mathbf{r}(w.\mathcal{D}) = \mathbf{r}(\mathcal{D})$ and $\mathbf{c}(w.\mathcal{D}) = (c_{w^{-1}(1)}(\mathcal{D}), \dots, c_{w^{-1}(N)}(\mathcal{D}))$, where \mathbf{r}, \mathbf{c} are the margin vectors from Subsection 2.1. Likewise, if $D \in \text{Tie}(\mathcal{D})$ we define $w.D \in \text{Tie}(w.\mathcal{D})$ via

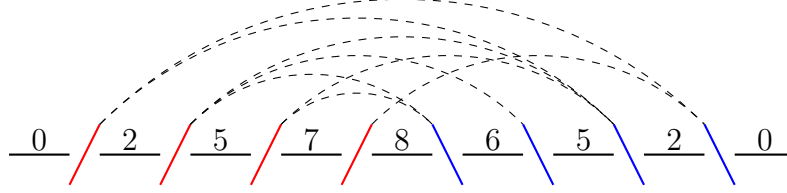
$$w.D = \bigcup_{(V_i, U_j) \in D} \{(V_i, U_{w(j)})\}.$$

Pictorially, the action is given by moving each blue line U_i with its attached ties to the position of $U_{w(i)}$.

Example 5.2. Consider the following tie diagram D with underlying brane diagram \mathcal{D} :



Let $w = 3142 \in S_4$. To obtain the tie diagram $w.D$, we permute the blue lines with the attached ties according to w , i.e. the blue line U_1 is moved with its three attached ties to the position of U_3 and so on. The respective labels of the horizontal lines of $w.D$ can then be easily determined by counting the number of ties above the horizontal lines:



Stable basis elements satisfy the following compatibility relation with this S_N -action, see [BR23, Proposition 6.18] and [Weh24, Theroem 9.20]:

Theorem 5.3. *For $D, D' \in \text{Tie}(\mathcal{D})$ and $w \in S_N$, we have*

$$e_{\mathbb{T}}(N_{\mathcal{D}, \mathfrak{C}}^-) \iota_{D'}^*(\text{Stab}_{\mathfrak{C}}(D)) = w^{-1} \cdot (e_{\mathbb{T}}(N_{w.\mathcal{D}, w.\mathfrak{C}}^-) \iota_{w.D'}^*(\text{Stab}_{w.\mathfrak{C}}(w.D))).$$

Here, $w \in S_N$ acts on $\mathbb{Q}[t_1, \dots, t_N, h]$ via $w.h = h$ and $w.t_i = t_{w(i)}$ for $i = 1, \dots, N$ and $N_{\mathcal{D}, \mathfrak{C}}^-$ (resp. $N_{w.\mathcal{D}, w.\mathfrak{C}}^-$) is the negative part of the trivial \mathbb{T} -equivariant bundle $N_{\mathcal{D}}$ (resp. $N_{w.\mathcal{D}}$) defined in the remark below.

Remark 5.4. Set

$$N_{\mathcal{D}} := \left(\bigoplus_{j=1}^N \bigoplus_{l=1}^{c_j-1} h^l (\xi_{U_j^+} \otimes \mathbb{C}_{U_j}^{\vee}) \right) \oplus \left(\bigoplus_{j=1}^N \bigoplus_{l=1}^{c_j-1} h^{1-l} (\mathbb{C}_{U_j} \otimes \xi_{U_j^+}^{\vee}) \right). \quad (5.2)$$

As explained in [BR23, Corollary 6.6], the bow variety $\mathcal{C}(\mathcal{D})$ can be \mathbb{T} -equivariantly embedded into the partial flag variety $T^*F(R_1, \dots, R_M; n)$. As shown in [Weh24, Corollary 9.53], the \mathbb{T} -equivariant K-theory class of the normal bundle of this embedding is a sum of classes of trivial bundles. Its positive (resp. negative) part with respect to any choice of chamber equals the positive (resp. negative) part of $[N_{\mathcal{D}}]$.

Recall from Corollary 2.14 that for all $U_j \in \mathbf{b}(\mathcal{D})$, we have an isomorphism of \mathbb{T} -equivariant vector bundles $\xi_{U_j^+} \cong \bigoplus_{i=j+1}^N \bigoplus_{l=0}^{c_i-1} h^{-l} \mathbb{C}_{U_i}$. Thus, the positive (resp. negative) part of $N_{\mathcal{D}}$ with respect to a choice of chamber \mathfrak{C} can be easily read off from the definition. For instance, if \mathfrak{C} equals the antidominant chamber \mathfrak{C}_- then $N_{\mathcal{D}, \mathfrak{C}_-}^{\pm}$ are given as follows:

Proposition 5.5. *We have*

$$N_{\mathcal{D}, \mathfrak{C}_-}^+ = \bigoplus_{j=1}^N \bigoplus_{i=j+1}^N \bigoplus_{l=1}^{c_j-1} \bigoplus_{k=0}^{c_i-1} h^{l-k} (\mathbb{C}_{U_i} \otimes \mathbb{C}_{U_j}^{\vee}), \quad N_{\mathcal{D}, \mathfrak{C}_-}^- = \bigoplus_{j=1}^N \bigoplus_{i=j+1}^N \bigoplus_{l=1}^{c_j-1} \bigoplus_{k=0}^{c_i-1} h^{k-l+1} (\mathbb{C}_{U_j} \otimes \mathbb{C}_{U_i}^{\vee}).$$

Proof. By Corollary 2.14, all weights of $\xi_{U_j^+} \otimes \mathbb{C}_{U_j}^{\vee}$ are non-negative. Hence, only the second summand of (5.2) contributes to $N_{\mathcal{D}}^-$. Then, inserting $\xi_{U_j^+} \cong \bigoplus_{i=j+1}^N \bigoplus_{l=0}^{c_i-1} h^{-l} \mathbb{C}_{U_i}$ proves the proposition. \square

Because of Theorem 5.3, it is sometimes more convenient to work with the following normalized version of stable basis elements:

Notation. With the above notation, we set

$$\widetilde{\text{Stab}}_{\mathfrak{e}}(D) := e_{\mathbb{T}}(N_{\mathcal{D}, \mathfrak{e}}^-) \text{Stab}_{\mathfrak{e}}(D), \quad \text{for all } D \in \text{Tie}(\mathcal{D}). \quad (5.3)$$

In the special case $\mathcal{D} = \mathcal{D}(d_1, \dots, d_m; n)$ is as in Subsection 3.5, we have $c_1 = \dots = c_n = 1$. Thus, by definition, $N_{\mathcal{D}}$ is the trivial bundle of rank zero which yields $\widetilde{\text{Stab}}_{\mathfrak{e}}(D) = \text{Stab}_{\mathfrak{e}}(D)$ in this case.

Remark 5.6. In [BR23, Proposition 6.18], Rimányi and Botta prove a version Theorem 5.3 in the framework of elliptic cohomology with a different normalization factor.

6. STABLE BASES OF COTANGENT BUNDLES OF FLAG VARIETIES

In this section, we recall the localization formula for stable basis elements of cotangent bundles of flag varieties from [Su17a, Theorem 1.1], see also [Su17b]. Then, we give an equivalent reformulation of this formula using the language of diagrammatic calculus of symmetric groups.

6.1. Reminders on symmetric groups. We denote the simple transpositions of S_n by s_1, \dots, s_{n-1} , where $s_i = (i, i+1)$. Every permutation can be written as $w = \sigma_1 \cdots \sigma_r$, where all σ_i are simple transpositions. If r is as small as possible, we call the expression $\sigma_1 \cdots \sigma_r$ for w *reduced* and we call r the *length* of w and denote it by $\ell(w)$.

It is well-known that $\ell(w)$ is equal to the number of inversions of w :

$$\ell(w) = |\text{Inv}(w)|, \quad \text{Inv}(w) = \{(i, j) \mid 1 \leq i < j \leq n, w(i) > w(j)\}. \quad (6.1)$$

By definition, a permutation w is larger than a permutation w' in the *Bruhat order* if some (not necessarily a consecutive) subword of a reduced expression for w is a reduced word for w' . It is a well-known fact that if w dominates w' in the Bruhat order then every reduced expression for w admits a subword which is a reduced expression for w' , see e.g. [Hum90, Theorem 5.10].

Let $R^+ = \{t_i - t_j \mid 1 \leq i < j \leq n\} \subset \mathbb{Q}[t_1, \dots, t_n]$ be the set of positive roots and $R^- = \{t_i - t_j \mid 1 \leq j < i \leq n\} \subset \mathbb{Q}[t_1, \dots, t_n]$ the set of negative roots. By (6.1), we have

$$\ell(w) = |\{\alpha \in R^+ \mid w.\alpha \in R^-\}|. \quad (6.2)$$

The set on the right hand side of (6.2) can also be characterized as follows: for $s = s_i$ we denote by $\alpha_s = t_i - t_{i+1}$ the corresponding simple root. Given a reduced expression $w = \sigma_1 \cdots \sigma_{\ell(w)}$, we set

$$\beta_i := \sigma_1 \cdots \sigma_{i-1}(\alpha_{\sigma_i}), \quad i = 1, \dots, \ell(w). \quad (6.3)$$

Then, by e.g. [Hum90, Section 5.6], we have

$$\{\beta_1, \dots, \beta_{\ell(w)}\} = \{\alpha \in R^+ \mid w^{-1}.\alpha \in R^-\}. \quad (6.4)$$

Example 6.1. Let $n = 5$ and $w = 35412$. Then, $\ell(w) = 7$ and $w = s_4 s_2 s_1 s_3 s_2 s_4 s_3 =: \sigma_1 \cdots \sigma_7$ is a reduced expression of w . The corresponding β_i are recorded in the following table:

i	1	2	3	4	5	6	7
β_i	$t_4 - t_5$	$t_2 - t_3$	$t_1 - t_3$	$t_2 - t_5$	$t_1 - t_5$	$t_2 - t_4$	$t_1 - t_4$

6.2. Diagrammatics of permutations. We illustrate permutations in the common way using string diagrams as follows: a *strand* is a smooth embedding $\lambda: [0, 1] \rightarrow \mathbb{R}^2$.

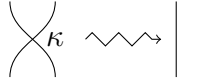
Definition 6.2. Let $w \in S_n$ be a permutation. A collection $\lambda_1, \dots, \lambda_n$ of n strands is called a *diagram of w* if the following holds:

- (i) $\lambda_i(0) = (i, 0)$ and $\lambda_i(1) = (w(i), 1)$ for all $i = 1, \dots, n$,
- (ii) every two strands intersect only in finitely many points and all of these intersections are transversal,
- (iii) there are no triple or even higher intersections among $\lambda_1, \dots, \lambda_n$.

A diagram is called *reduced* if the number of intersections among $\lambda_1, \dots, \lambda_n$ is equal to $\ell(w)$.

We call the intersection of two strands a *crossing*. Given a diagram d_w of a permutation w , we denote by $K(d_w)$ its set of crossings. If the second coordinates of all crossings of d_w are pairwise distinct, we denote the crossings of d_w by $\kappa_1, \dots, \kappa_{\ell(w)}$ where κ_1 is the crossing with the highest second coordinate, κ_2 is the crossing with the second highest second coordinate and so on.

Given a crossing $\kappa \in K(d_w)$ we refer to the local move

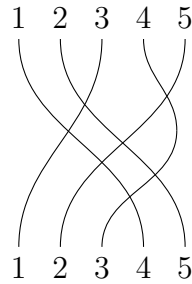


as *resolving* of κ .

By definition, w is larger than a permutation $w' \in S_n$ in the Bruhat order if and only if we can obtain a diagram for w' by resolving crossing from d_w .

If d_w is a reduced diagram such that all crossings of d_w have pairwise distinct second coordinate then, by viewing d_w as stacking of diagrams corresponding to simple transpositions, we can easily read off a reduced expression for w from d_w .

Example 6.3. Let $w \in S_5$ be as in Example 6.1. Then, a reduced diagram d_w of w is given by



The diagram corresponds to the reduced expression $s_4 s_2 s_1 s_3 s_2 s_4 s_3$.

For given $w \in S_n$ with reduced diagram d_w , define a function

$$\text{wt}: K(d_w) \longrightarrow \mathbb{Q}[t_1, \dots, t_n]$$

as follows: let κ be a crossing between the strands λ and λ' . Let j resp. j' be the endpoints of λ resp. λ' . Assuming $j < j'$, we set

$$\text{wt}(\kappa) := t_j - t_{j'}.$$

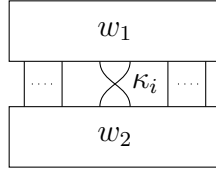
We call $\text{wt}(\kappa)$ the *weight* of κ .

The next proposition gives that the weights of crossings correspond exactly to the β_i from the previous subsection:

Proposition 6.4. *Suppose all crossings of d_w have pairwise distinct second coordinate and let $w = \sigma_1 \cdots \sigma_{\ell(w)}$ be the reduced expression corresponding to d_w . Then, we have*

$$\text{wt}(\kappa_i) = \beta_i, \quad \text{for all } i = 1, \dots, \ell(w).$$

Proof. For given $i \in \{1, \dots, \ell(w)\}$, set $w_1 := \sigma_1 \cdots \sigma_{i-1}$, $w_2 := \sigma_{i+1} \cdots \sigma_{\ell(w)}$ and $\sigma_i = (j, j+1)$. After applying a homotopy, we can view d_w as a stacking of a reduced diagram of w_2 , a reduced diagram of σ_i and a reduced diagram of w_1 :



Thus, we have $\text{wt}(\kappa_i) = t_{w_1(j)} - t_{w_1(j+1)} = w_1 \cdot \alpha_{\sigma_i} = \beta_i$ which completes the proof. \square

6.3. Localization formula. Let $F = F(1, 2, \dots, n-1; n)$ be the full flag variety of \mathbb{C}^n endowed with the \mathbb{T} -action from Subsection 3. For $w \in S_n$, we also denote the \mathbb{T} -fixed point $(\mathcal{F}_w, 0)$ just by w .

The localization formula from [Su17a, Theorem 1.1] determines the \mathbb{T} -localization coefficients of the stable basis elements of T^*F with respect to the antidominant chamber \mathfrak{C}_- .

Theorem 6.5 (Localization formula). *Let $w \in S_n$ and $w = \sigma_1 \sigma_2 \cdots \sigma_{\ell(w)}$ a reduced expression. Then, for all $w' \in S_n$, we have*

$$\iota_w^*(\text{Stab}_{\mathfrak{C}_-}(w')) = \left(\prod_{(i,j) \in L_w} (t_i - t_j + h) \right) \left(\sum_{\substack{1 \leq i_1 < \dots < i_k \leq \ell(w) \\ w' = \sigma_{i_1} \cdots \sigma_{i_k}}} h^{\ell(w)-k} \prod_{j=1}^k \beta_{i_j} \right),$$

where the β_i are defined as in (6.3) and

$$L_w = R^+ \setminus \{\alpha \in R^+ \mid w^{-1} \cdot \alpha \in R^-\} = \{\alpha \in R^+ \mid \alpha \neq \beta_l \text{ for all } l\}.$$

Example 6.6. Let $n = 5$ and consider the permutations $w = 35412$ and $w' = 23415$. To compute the localization coefficient $\iota_w^*(\text{Stab}_{\mathfrak{C}_-}(w'))$, we choose, as in Example 6.1, $w = s_4 s_2 s_1 s_3 s_2 s_4 s_3$ as reduced expression for w . Checking all possible subwords of this expression for w yields that there are only two subwords that give w' , namely $\sigma_1 \sigma_3 \sigma_5 \sigma_6 \sigma_7$ and $\sigma_3 \sigma_5 \sigma_7$. We already computed the β_i in Example 6.1. Our computation implies $L_w = \{(t_1 - t_2), (t_3 - t_4), (t_3 - t_5)\}$. Hence, Theorem 6.5 yields

$$\iota_w^*(\text{Stab}_{\mathfrak{C}_-}(w')) = (t_1 - t_2 + h)(t_3 - t_4 + h)(t_3 - t_5 + h) \cdot h^2(\beta_1 \beta_6 + h^2) \beta_3 \beta_5 \beta_7. \quad (6.5)$$

6.4. Diagrammatic localization formula. Employing the diagrammatics related to permutations from Subsection 6.2 leads to the following diagrammatic version of Theorem 6.5:

Proposition 6.7 (Diagrammatic localization formula). *Let $w \in S_n$ and d_w a reduced diagram of w . Then, for all $w' \in S_n$, we have*

$$\iota_w^*(\text{Stab}_{\mathfrak{C}_-}(w')) = \left(\prod_{\alpha \in L'_w} (\alpha + h) \right) \left(\sum_{K' \in K_{d_w, w'}} h^{|K'|} \prod_{\kappa \in K(d_w) \setminus K'} \text{wt}(\kappa) \right),$$

where $K_{d_w, w'}$ is the set of all subsets $K' \subset K(d_w)$ such that resolving all crossings of K' from d_w gives a diagram for w' and

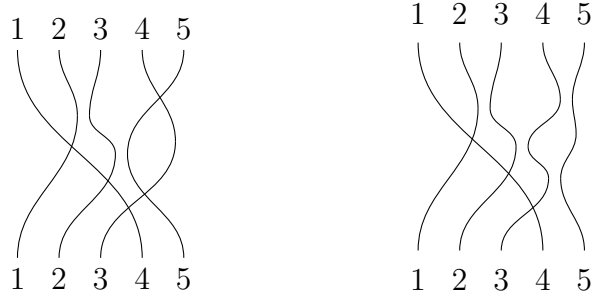
$$L'_w = \{\alpha \in R^+ \mid \alpha \neq \text{wt}(\kappa) \text{ for all } \kappa \in K(d_w)\}.$$

Proof. We may assume without loss of generality that the second coordinates of all crossings in d_w are pairwise distinct. Let $1 \leq i_1 < \dots < i_k \leq \ell(w)$ and let d' be the diagram obtained from d_w by resolving all crossings κ_i with $i \neq i_1, \dots, i_k$. By viewing d_w as a stacking of diagrams corresponding to simple transpositions we deduce that $w' = \sigma_{i_1} \dots \sigma_{i_k}$ if and only if d' is a diagram for w' . Thus, Proposition 6.4 implies

$$\sum_{K' \in K_{d_w, w'}} h^{|K'|} \prod_{\kappa \in K(d_w) \setminus K'} \text{wt}(\kappa) = \sum_{\substack{1 \leq i_1 < \dots < i_k \leq \ell(w) \\ w' = \sigma_{i_1} \dots \sigma_{i_k}}} h^{\ell(w) - k} \prod_{j=1}^k \beta_{i_j}.$$

In addition, Proposition 6.4 also gives $L_w = L'_w$ which completes the proof. \square

Example 6.8. Let w and w' be as in Example 6.6 and let d_w be as in Example 6.3. To compute $\iota_w^*(\text{Stab}_{\mathfrak{C}_-}(w'))$, note that there are just two possibilities to obtain a diagram for w' by resolving crossings from d_w . One is given by resolving the crossings κ_2 and κ_4 . The second one is given by resolving the crossings $\kappa_1, \kappa_2, \kappa_4$ and κ_6 , in pictures:



The diagram on the left corresponds to the subword $\sigma_1\sigma_3\sigma_5\sigma_6\sigma_7$ and the diagram on the right hand side to $\sigma_3\sigma_5\sigma_7$. The first diagram contributes the summand

$$(t_1 - t_2 + h)(t_3 - t_4 + h)(t_3 - t_5 + h)h^2 \text{wt}(\kappa_1) \text{wt}(\kappa_3) \text{wt}(\kappa_5) \text{wt}(\kappa_6) \text{wt}(\kappa_7),$$

whereas the second diagram contributes the summand

$$(t_1 - t_2 + h)(t_3 - t_4 + h)(t_3 - t_5 + h)h^4 \text{wt}(\kappa_3) \text{wt}(\kappa_5) \text{wt}(\kappa_7).$$

It follows that

$$\begin{aligned} \iota_w^*(\text{Stab}_{\mathfrak{C}_-}(w')) = \\ (t_1 - t_2 + h)(t_3 - t_4 + h)(t_3 - t_5 + h) \cdot h^2(\text{wt}(\kappa_1) \text{wt}(\kappa_6) + h^2) \text{wt}(\kappa_3) \text{wt}(\kappa_5) \text{wt}(\kappa_7) \end{aligned}$$

which coincides with the computation (6.5) from Example 6.6.

6.5. Cotangent bundles of partial flag varieties. Let $F = F(d_1, \dots, d_m; n)$ be a partial flag variety and $\delta = (\delta_1, \dots, \delta_{m+1})$ as in Subsection 3. As before, for a given $w \in S_n$, we also denote the \mathbb{T} -fixed point $(\mathcal{F}_{wS_\delta}, 0)$ by wS_δ .

It was proved in [Su17a, Corollary 4.3] that the localization coefficients of the stable basis elements of T^*F can be computed via localization coefficients of stable basis elements of $T^*F(1, 2, \dots, n-1; n)$ as follows:

Proposition 6.9. *For all $w, w' \in S_n$, we have*

$$\iota_{wS_\delta}^*(\text{Stab}_{\mathfrak{C}_-}(w'S_\delta)) = \sum_{z \in wS_\delta} \frac{(-1)^{\ell(w'S_\delta) + \ell(w')} \iota_z^*(\text{Stab}_{\mathfrak{C}_-}(w'))}{\prod_{\alpha \in R_\delta} (z \cdot \alpha)},$$

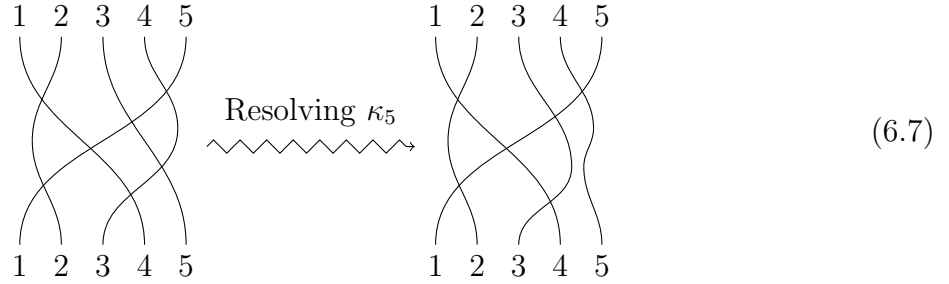
where $\ell(wS_\delta)$ is the length of the shortest coset representative of wS_δ and

$$R_\delta^+ = \{t_i - t_j \mid \text{there exist } l \in \{1, \dots, r\} \text{ with } d_1 + \dots + d_{l-1} \leq i < j \leq d_1 + \dots + d_l\}.$$

Example 6.10. Let $\delta = (2, 2, 1)$ and $w = 25143$, $w' = 52314$. Note that w is the minimal element of wS_δ . Using that $\ell(w) = 5$ and $\ell(w') = 6$, we deduce that $z = w(s_1 \times s_1 \times \text{id})$ is the only element in wS_δ that dominates w' in the Bruhat order. Hence, Proposition 6.9 gives

$$\iota_{wS_\delta}^*(\text{Stab}_{\mathfrak{C}_-}(w'S_\delta)) = \frac{\iota_z^*(\text{Stab}_{\mathfrak{C}_-}(w'))}{(t_{z(1)} - t_{z(2)})(t_{z(3)} - t_{z(4)})} = \frac{\iota_z^*(\text{Stab}_{\mathfrak{C}_-}(w'))}{(t_5 - t_2)(t_4 - t_1)}. \quad (6.6)$$

To compute $\iota_z^*(\text{Stab}_{\mathfrak{C}_-}(w'))$ we use Proposition 6.7. The following figure shows a reduced diagram d_z for z . Since $\ell(z) = 7$ and $\ell(w') = 6$, there is only one possibility to obtain a diagram $d_{w'}$ for w' from d_z by resolving crossings:



We record the weights of the crossings of d_z in the following table:

i	1	2	3	4	5	6	7
$\text{wt}(\kappa_i)$	$t_4 - t_5$	$t_1 - t_2$	$t_3 - t_5$	$t_1 - t_5$	$t_3 - t_4$	$t_2 - t_5$	$t_1 - t_4$

Thus, we have

$$\iota_z^*(\text{Stab}_{\mathfrak{C}_-}(w')) = (t_1 - t_3 + h)(t_2 - t_3 + h)(t_2 - t_4 + h)h \cdot \prod_{i \neq 5} \text{wt}(\kappa_i)$$

which implies

$$(6.6) = (t_1 - t_3 + h)(t_2 - t_3 + h)(t_2 - t_4 + h)h(t_4 - t_5)(t_1 - t_2)(t_3 - t_5)(t_1 - t_5). \quad (6.8)$$

6.6. Transition invariance. Let $\mathcal{D}(d_1, \dots, d_m; n)$ be the separated brane diagram from Subsection 3 and $\Phi \circ \Psi: F(d_1, \dots, d_m; n) \xrightarrow{\sim} \mathcal{C}(\mathcal{D}(d_1, \dots, d_m; n))$ be the isomorphism from (3.1) and (3.4). By Proposition 2.15, this isomorphism is ρ -equivariant, where

$$\rho: \mathbb{T} \xrightarrow{\sim} \mathbb{T}, \quad (t_1, \dots, t_n, h) \mapsto (t_1 h, \dots, t_n h, h).$$

Thus, by Proposition 4.6, we have

$$\iota_{D_{wS_\delta}}^*(\text{Stab}_{\mathfrak{C}_-}(D_{w'S_\delta})) = \varphi(\iota_{wS_\delta}^*(\text{Stab}_{\mathfrak{C}_-}(w'S_\delta))), \quad \text{for all } w, w' \in S_n. \quad (6.9)$$

Here $\varphi: \mathbb{Q}[t_1, \dots, t_n, h] \xrightarrow{\sim} \mathbb{Q}[t_1, \dots, t_n, h]$ is the $\mathbb{Q}[h]$ -algebra automorphism given by $t_i \mapsto t_i - h$ for all i .

The localization formula implies that the localization coefficients of stable basis elements of $\mathcal{C}(\mathcal{D}(d_1, \dots, d_m; n))$ are φ -invariant:

Proposition 6.11. *We have for all $w, w' \in S_n$*

$$\iota_{D_{wS_\delta}}^*(\text{Stab}_{\mathfrak{C}_-}(D_{w'S_\delta})) = \iota_{wS_\delta}^*(\text{Stab}_{\mathfrak{C}_-}(w'S_\delta)).$$

Proof. Since we have $\varphi(t_i - t_j + mh) = t_i - t_j + mh$ for all $i, j \in \{1, \dots, n\}$ and $m \in \mathbb{Z}$, Theorem 6.5 and Proposition 6.9 imply

$$\varphi(\iota_{wS_\delta}^*(\text{Stab}_{\mathfrak{C}_-}(w'S_\delta))) = \iota_{wS_\delta}^*(\text{Stab}_{\mathfrak{C}_-}(w'S_\delta)).$$

Hence, the proposition follows from (6.9). \square

7. SYMMETRIC GROUP CALCULUS FOR BOW VARIETIES

Let \mathcal{D} be a brane diagram and recall the margin vectors \mathbf{r} and \mathbf{c} from Subsection 2.6. Let $n = \sum_{i=1}^M r_i = \sum_{j=1}^N c_j$. We denote by

$$S_{\mathbf{c}} := S_{c_1} \times \dots \times S_{c_N} \subset S_n \quad \text{and} \quad S_{\mathbf{r}} := S_{r_1} \times \dots \times S_{r_M} \subset S_n$$

the corresponding Young subgroups.

In this section, we describe a correspondence between the binary contingency tables of \mathcal{D} and a special class of $(S_{\mathbf{c}}, S_{\mathbf{r}})$ -double cosets which we call *fully separated*, see Definition 7.2. As we will discuss in Subsection 7.3, permutations that belong to fully separated double cosets satisfy strong uniqueness properties which distinguish fully separated double cosets from general double cosets.

7.1. Fully separated double cosets. The usual assignment of a $(S_{\mathbf{c}}, S_{\mathbf{r}})$ -double coset to a matrix leads to the following well-known bijection, see e.g. [JK81, Theorem 1.3.10]:

Theorem 7.1. *Let $\Xi(\mathbf{r}, \mathbf{c})$ be the set of all $M \times N$ -matrices A with entries in $\mathbb{Z}_{\geq 0}$ satisfying*

$$\sum_{l=1}^N A_{i,l} = r_i, \quad \sum_{l=1}^M A_{l,j} = c_j, \quad \text{for all } i, j.$$

Then, the map $Z: S_n \rightarrow \Xi(\mathbf{c}, \mathbf{r})$ given by

$$Z(w)_{i,j} = |w(\{R_{i-1} + 1, \dots, R_i\}) \cap \{C_{j-1} + 1, \dots, C_j\}|$$

induces a bijection

$$\bar{Z}: S_{\mathbf{c}} \backslash S_n / S_{\mathbf{r}} \xrightarrow{\sim} \Xi(\mathbf{r}, \mathbf{c}), \quad S_{\mathbf{c}} w S_{\mathbf{r}} \mapsto Z(w). \quad (7.1)$$

By definition, the elements of $\text{bct}(\mathcal{D})$ are exactly the matrices in $\Xi(\mathbf{r}, \mathbf{c})$ with all entries contained in $\{0, 1\}$. The following notion characterizes the double cosets that correspond to $\text{bct}(\mathcal{D})$ under \bar{Z} :

Definition 7.2. A permutation $w \in S_n$ is called *fully separated (with respect to (\mathbf{r}, \mathbf{c}))* if

$$|w(\{R_{i-1} + 1, \dots, R_i\}) \cap \{C_{j-1} + 1, \dots, C_j\}| \leq 1,$$

for all $i \in \{1, \dots, M\}$, $j \in \{1, \dots, N\}$.

If w is fully separated then so is every element in $S_{\mathbf{c}}wS_{\mathbf{r}}$. Hence, we call a double coset $S_{\mathbf{c}}wS_{\mathbf{r}}$ fully separated if all its elements are fully separated. Likewise, we call a left $S_{\mathbf{r}}$ -coset (resp. right $S_{\mathbf{c}}$ -coset) fully separated if all its elements are fully separated.

Clearly, a permutation w is fully separated if and only if $Z(w)$ is contained in $\text{bct}(\mathcal{D})$. Thus, we have the following corollary:

Corollary 7.3. *The bijection \bar{Z} from (7.1) restricts to a bijection*

$$\text{fsep}_{\mathbf{c},\mathbf{r}} \xrightarrow{\sim} \text{bct}(\mathcal{D}),$$

where $\text{fsep}_{\mathbf{c},\mathbf{r}}$ denotes the set of fully separated $(S_{\mathbf{c}}, S_{\mathbf{r}})$ -double cosets.

Example 7.4. Let $n = 5$ and $\mathbf{r} = (2, 2, 1)$, $\mathbf{c} = (1, 2, 2)$. The permutations $w_1 = 14253$ and $w_2 = 14235$ get assigned to the matrices

$$Z(w_1) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad Z(w_2) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus w_1 is fully separated, as all entries of $Z(w_1)$ are contained in $\{0, 1\}$. On the other hand, $Z(w_2)$ admits an entry equal to 2, so w_2 is not fully separated.

In diagrammatic language the fully separatedness condition can be reformulated as follows: given $w \in S_N$ and a diagram d_w for w . Then, w is fully separated if and only if for all $i \in \{1, \dots, M\}$, $j \in \{1, \dots, N\}$, there exists at most one strand with source in $\{R_{i-1} + 1, \dots, R_i\}$ and target in $\{C_{j-1} + 1, \dots, C_j\}$.

Remark 7.5. In [JK81] the fully separatedness condition is called *trivial intersection property*.

7.2. Shortest double coset representatives. It is well-known, see e.g. [Hum90, Section 5.12], that each left coset $wS_{\mathbf{r}}$ (resp. right coset $S_{\mathbf{c}}w$) contains a unique representative of shortest Bruhat length w_l (resp. w_r) which is uniquely determined by the condition $w_l(R_{i-1} + 1) < \dots < w_l(R_i)$ for all i (resp. $w_r^{-1}(C_{j-1} + 1) < \dots < w_r^{-1}(C_j)$ for all j). Likewise, each double coset $S_{\mathbf{c}}wS_{\mathbf{r}}$ contains a unique representative of shortest Bruhat length w_d which is uniquely characterized by the conditions

$$w_d(R_{i-1} + 1) < \dots < w_d(R_i) \quad \text{and} \quad w_d^{-1}(C_{j-1} + 1) < \dots < w_d^{-1}(C_j), \quad \text{for all } i, j.$$

In the following, we describe the shortest representative of $(S_{\mathbf{c}}, S_{\mathbf{r}})$ -double cosets corresponding to binary contingency tables. We begin with a hopefully intuitive example:

Example 7.6. Let $n = 10$, $\mathbf{r} = (3, 2, 2, 3)$, $\mathbf{c} = (2, 3, 2, 1, 2)$ and

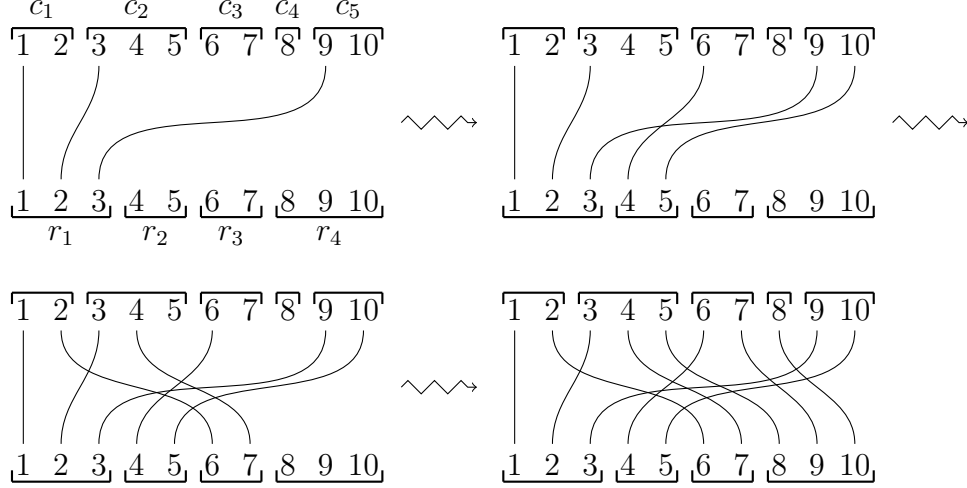
$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

We draw a diagram for the shortest double coset representative of $\bar{Z}^{-1}(A)$ following the next steps: at first, we define functions

$$F_{A,i}: \{1, \dots, r_i\} \longrightarrow \{1, \dots, N\}, \quad i = 1, \dots, M,$$

where $F_{A,i}(l)$ is the column index of the l -th 1-entry in the i -th row of A . For instance, $F_{A,1}: \{1, 2, 3\} \rightarrow \{1, \dots, 5\}$ is given by $F_{A,1}(1) = 1$, $F_{A,1}(2) = 2$ and $F_{A,1}(3) = 5$.

We start drawing our diagram by drawing strands λ_l starting in $l = 1, \dots, r_1$ and ending in $C_{F_{A,1}(1)-1} + 1, \dots, C_{F_{A,1}(r_1)-1} + 1$. Then, we draw strands λ_{r_1+l} starting in $r_1 + 1, \dots, r_1 + r_2$ and the endpoint of λ_{r_1+l} is the smallest element of $\{C_{F_{A,2}(l)-1} + 1, \dots, C_{F_{A,2}(l)}\}$ that is not already the endpoint of a strand. Continuing this procedure leads to the following permutation diagram:



We denote the resulting permutation by \tilde{w}_A , i.e. $\tilde{w}_A = 13961024578$. Our condition to pick always the smallest entry in $\{C_{j-1} + 1, \dots, C_j\}$ that is not already the endpoint of a strand implies $\tilde{w}_A^{-1}(C_{j-1} + 1) < \dots < \tilde{w}_A^{-1}(C_j)$, for all j . In addition, as the functions $F_{A,i}$ strictly increase, we also have $\tilde{w}_A(R_{i-1} + 1) < \dots < \tilde{w}_A(R_i)$, for all i . Thus, \tilde{w}_A is a shortest (S_c, S_r) -double coset representative. As \tilde{w}_A satisfies $\tilde{w}_A(R_{i-1} + l) \in \{C_{F_{A,i}(l)-1} + 1, \dots, C_{F_{A,i}(l)}\}$ for all i, l , we conclude

$$Z(\tilde{w}_A)_{R_{i-1}+l,j} = \begin{cases} 1 & \text{if } j = F_{A,i}(l), \\ 0 & \text{if } j \neq F_{A,i}(l). \end{cases}$$

Therefore, $Z(\tilde{w}_A) = A$ which implies that \tilde{w}_A is indeed the shortest representative of $\bar{Z}^{-1}(A)$.

We return to the general setup: let \mathcal{D} be a brane diagram and $A \in \text{bct}(\mathcal{D})$.

As in the previous example, let $F_{A,i}: \{1, \dots, r_i\} \rightarrow \{1, \dots, N\}$ be the function assigning to l the column index of the l -th 1-entry in the i -th row of A . Likewise, let $G_{A,j}: \{1, \dots, c_j\} \rightarrow \{1, \dots, M\}$ be the function assigning to l the row index of the l -th 1-entry in the j -th column of A . Furthermore, we set $n_{A,i,j} = \sum_{l=1}^i A_{l,j}$. That is, $n_{A,i,j}$ is the number of 1-entries that are in the j -th column of A and strictly above the entry $A_{i+1,j}$.

Definition 7.7. We define the permutation $\tilde{w}_A \in S_n$ as

$$\tilde{w}_A(R_{i-1} + l) = C_{F_{A,i}(l)-1} + n_{A,i,F_{A,i}(l)}, \quad \text{for } i = 1, \dots, M \text{ and } l = 1, \dots, r_i.$$

To see that \tilde{w}_A is a permutation, note that if we are given $j \in \{1, \dots, N\}$ and $l \in \{1, \dots, c_j\}$ then let $i = G_{A,j}(l)$ and $l' \in \{1, \dots, r_i\}$ such that $A_{i,j}$ is the l' -th 1-entry in the i -th row of A . By construction, we have $\tilde{w}_A(R_{i-1} + l') = C_{j-1} + l$ which proves the surjectivity of \tilde{w}_A . Hence, \tilde{w}_A is a permutation.

The next proposition shows that \tilde{w}_A satisfies the desired properties.

Proposition 7.8. *The following holds:*

- (i) $Z(\tilde{w}_A) = A$,
- (ii) \tilde{w}_A is the shortest representative of $\bar{Z}^{-1}(A)$,
- (iii) we have $\ell(\tilde{w}_A) = |\text{Inv}(A)|$, where

$$\text{Inv}(A) = \{((i_1, j_1), (i_2, j_2)) \mid A_{i_1, j_1} = A_{i_2, j_2} = 1, i_1 < i_2, j_2 < j_1\}.$$

Proof. By construction, $\tilde{w}_A(R_{i-1} + l) \in \{C_{F_{A,i}(l)-1} + 1, \dots, C_{F_{A,i}(l)}\}$ which gives

$$Z(\tilde{w}_A)_{R_{i-1}+l, j} = \begin{cases} 1 & \text{if } j = F_{A,i}(l), \\ 0 & \text{otherwise.} \end{cases}$$

Hence, $Z(\tilde{w}_A) = A$. Moreover, we conclude $\tilde{w}_A(R_{i-1} + 1) < \dots < \tilde{w}_A(R_i)$ for all i . The explicit description of \tilde{w}_A above gives $\tilde{w}_A^{-1}(C_{j-1} + l) \in \{R_{G_{A,j}(l)-1} + 1, \dots, R_{G_{A,j}(l)}\}$ which implies $\tilde{w}_A^{-1}(C_{j-1} + 1) < \dots < \tilde{w}_A^{-1}(C_j)$ for all j . Thus, \tilde{w}_A is the shortest representative of $\bar{Z}^{-1}(A)$. Finally, note that since \tilde{w}_A is a shortest left $S_{\mathbf{r}}$ -coset representative, the inversions of \tilde{w}_A are exactly the ordered pairs $(R_{i_1} + l_1, R_{i_2} + l_2)$ with

$$1 \leq i_1 < i_2 \leq M, \quad 1 \leq l_1 \leq r_{i_1}, \quad 1 \leq l_2 \leq r_{i_2}, \quad F_{A,i_1}(l_1) > F_{A,i_2}(l_2).$$

It follows that we have a bijection $\text{Inv}(\tilde{w}_D) \xrightarrow{\sim} \text{Inv}(D)$, where an inversion $(R_{i_1} + l_1, R_{i_2} + l_2)$ of \tilde{w}_A is mapped to $((i_1, F_{D,i_1}(l_1)), (i_2, F_{D,i_2}(l_2)))$. Hence, $\ell(\tilde{w}_D) = |\text{Inv}(D)|$. \square

Example 7.9. Let $w, w' \in S_5$ be as in Example 6.10 and choose $\mathbf{r} = (2, 2, 1)$, $\mathbf{c} = (2, 1, 2)$. Then, we have

$$Z(w) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad Z(w') = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We leave it as an exercise to the reader to that $\tilde{w}_{Z(w)} = 14253$ and $\tilde{w}_{Z(w')} = 14235$.

7.3. Uniqueness properties. In this subsection, we discuss strong uniqueness properties of fully separated permutations that distinguish them from general permutations. The central result is the following proposition:

Proposition 7.10. *Assume $w \in S_n$ is fully separated. Let $v, v' \in S_{\mathbf{r}}$ and $u, u' \in S_{\mathbf{c}}$ such that $uwv = u'wv'$. Then, $u = u'$ and $v = v'$.*

Before we prove Proposition 7.10, we illustrate the idea of the proof in the following example:

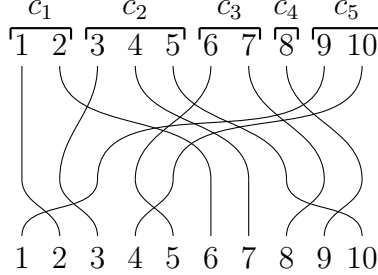
Example 7.11. Let $n, \mathbf{r}, \mathbf{c}$ and A be as in Example 7.6. For a permutation $w \in S_n$, we define the function

$$F_w: \{1, \dots, n\} \longrightarrow \{1, \dots, N\} \quad i \mapsto F_w(i),$$

where $F_w(i)$ is the unique element in $\{1, \dots, N\}$ such that $C_{F_w(i)-1} + 1 \leq w(i) \leq C_{F_w(i)}$. In terms of diagrammatic calculus, the function F_w can be characterized as follows: pick a diagram for w . Then, on the top, draw N square brackets around the intervals $\{1, \dots, C_1\}, \{C_1 + 1, \dots, C_2\}, \dots, \{C_{N-1} + 1, \dots, C_N\}$ and label them with $1, \dots, N$ from left to right. Then,

$F_w(i)$ is the index of the square bracket containing the endpoint of the unique strand starting in i .

Let for instance $v = v_1 \times v_2 \times v_3 \times v_4 \in S_r$, where $v_1 = 312$, $v_2 = 21$, $v_3 = 12$, $v_4 = 231$. A diagram for $\tilde{w}_A v$ is given by



The functions $F_{\tilde{w}_A}$ and $F_{\tilde{w}_A v}$ can be easily read off from their diagrams:

i	1	2	3	4	5	6	7	8	9	10
$F_{\tilde{w}_A}(i)$	1	2	5	3	5	1	2	2	3	4
$F_{\tilde{w}_A v}(i)$	5	1	2	5	3	1	2	3	4	5

Next, we show that if we know $F_{\tilde{w}_A v}$ then we can reconstruct the permutation v . We begin with reconstructing the factor v_1 . The first three letters in the row of $F_{\tilde{w}_A v}$ give the word 512 then using the identification $1 \mapsto 1$, $2 \mapsto 2$, $5 \mapsto 3$, we see that 512 corresponds to $312 = v_1$. Next, the fourth and the fifth letters in the row of $F_{\tilde{w}_A v}$ give the word 53. Using the identification $3 \mapsto 1$, $5 \mapsto 2$, we get the word $21 = v_2$. In the same way one can construct v_3 and v_4 and thus the permutation v .

In our reasoning, the fully separatedness property was essential because this property ensures that the restriction of $F_{\tilde{w}_A v}$ to $\{1, 2, 3\}$, $\{4, 5\}$, $\{6, 7\}$, $\{8, 9, 10\}$ is injective.

We proceed with the general setup. As in Example 7.11 we define for given $w \in S_n$ the function

$$F_w: \{1, \dots, n\} \longrightarrow \{1, \dots, N\}, \quad i \mapsto F_w(i), \quad (7.2)$$

where $F_w(i)$ is the unique element on $\{1, \dots, N\}$ such that

$$C_{F_w(i)-1} + 1 \leq w(i) \leq C_{F_w(i)}.$$

Likewise, we define

$$G_w: \{1, \dots, n\} \longrightarrow \{1, \dots, M\}, \quad j \mapsto G_w(j),$$

where $G_w(j)$ is the unique element on $\{1, \dots, M\}$ such that

$$R_{G_w(j)-1} + 1 \leq w^{-1}(j) \leq R_{G_w(j)}.$$

Similarly as F_w , the function G_w admits the following diagrammatic interpretation: pick a diagram for w and draw M square brackets on the bottom around the discrete intervals $\{1, \dots, R_1\}, \{R_1 + 1, \dots, R_2\}, \dots, \{R_{M-1} + 1, \dots, R_M\}$. Label the square brackets with $1, \dots, M$ from left to right. Then, $G_w(j)$ is the index of the square bracket containing the starting point of the unique strand with endpoint j .

If w is fully separated then the restrictions of F_w to the sets $\{R_{i-1} + 1, \dots, R_i\}$ is injective for $i = 1, \dots, M$. Likewise, the restriction of G_w to $\{C_{j-1} + 1, \dots, C_j\}$ is injective for $j = 1, \dots, N$. Moreover, the following properties are satisfied:

Lemma 7.12. *Assume $w \in S_n$ is fully separated. Given $u \in S_c$ and $v \in S_r$ then we have*

- (i) $F_{uw} = F_w$,
- (ii) $G_{wv} = G_w$,
- (iii) $F_{wv} = F_w$ if and only if $v = \text{id}$,
- (iv) $G_{uw} = G_w$ if and only if $u = \text{id}$.

Proof. Since u leaves the sets $\{C_{j-1} + 1, \dots, C_j\}$ invariant, we get (i). Likewise, v leaves the sets $\{R_{i-1} + 1, \dots, R_i\}$ invariant which gives (ii). For (iii), suppose that $v \neq \text{id}$ and $F_{wv} = F_w$. Let $l \in \{1, \dots, n\}$ such that $v(l) \neq l$. Choose $i \in \{1, \dots, M\}$ such that $l \in \{R_{i-1} + 1, \dots, R_i\}$. As $v \in S_r$, we have that $v(l)$ is also contained in $\{R_{i-1} + 1, \dots, R_i\}$. In addition, we have by definition $F_{wv}(l) = F_w(v(l))$ and hence $F_w(v(l)) = F_w(l)$ which contradicts the fact that the restriction of F_w to $\{R_{i-1} + 1, \dots, R_i\}$ is injective. The proof of (iv) is analogous. \square

Proof of Proposition 7.10. Without loss of generality, $u' = \text{id}$, $v' = \text{id}$. By Lemma 7.12.(i), $F_w = F_{u'wv'} = F_{wv}$. Thus, Lemma 7.12.(iii) implies $v = \text{id}$. Likewise, Lemma 7.12.(ii) gives $G_w = G_{u'wv'} = G_{uw}$ which implies $u = \text{id}$ by Lemma 7.12.(iv). \square

Given $A \in \text{bct}(\mathcal{D})$ then by construction, we have

$$F_{\tilde{w}_A}(R_{i-1} + l) = F_{A,i}(l), \quad \text{for } i = 1, \dots, M, l = R_{i-1} + 1, \dots, R_i. \quad (7.3)$$

This observation combined with Lemma 7.12 leads to the following characterization of shortest representatives of fully separated left resp. right cosets:

Corollary 7.13. *Given $A \in \text{bct}(\mathcal{D})$ then the following holds:*

- (i) $u\tilde{w}_A$ is a shortest left S_r -coset representative for all $u \in S_c$,
- (ii) if wS_r is a fully separated left coset then there exist $A \in \text{bct}(\mathcal{D})$, $u \in S_c$ such that $u\tilde{w}_A$ is the shortest representative of wS_r ,
- (iii) \tilde{w}_Av is a shortest right S_c -coset representative for all $v \in S_r$,
- (iv) if S_cw is a fully separated right coset then there exist $A \in \text{bct}(\mathcal{D})$, $v \in S_r$ such that \tilde{w}_Av is the shortest representative of S_cw .

Proof. By Lemma 7.12.(i) and (7.3), we have

$$F_{u\tilde{w}_A}(R_{i-1} + 1) < F_{u\tilde{w}_A}(R_{i-1} + 2) < \dots < F_{u\tilde{w}_A}(R_i), \quad \text{for } i = 1, \dots, M.$$

This implies $u\tilde{w}_A(R_{i-1} + 1) < u\tilde{w}_A(R_{i-1} + 2) < \dots < u\tilde{w}_A(R_i)$ for all i . Hence, $u\tilde{w}_A$ is the shortest representative of $u\tilde{w}_AS_r$ which gives (i). For (ii), we use that if wS_r is fully separated then $Z(w) \in \text{bct}(\mathcal{D})$ and there exist $u \in S_c$, $v \in S_r$ such that $w = u\tilde{w}_{Z(w)}v$. Thus, (i) gives that $u\tilde{w}_{Z(w)}$ is the shortest representative of wS_r . The proofs of (iii) and (iv) are analogous. \square

8. EQUIVARIANT RESOLUTION THEOREM

In this section, we discuss a version of the equivariant resolution theorem for bow varieties of Botta and Rimányi [BR23, Theorem 6.13]. This theorem allows to express localization coefficients of stable basis elements of bow varieties in terms of localization coefficients of stable basis elements of cotangent bundles of partial flag varieties.

We begin with describing the underlying combinatorics.

8.1. Resolutions of brane and tie diagrams. Let \mathcal{D} be an essential separated brane diagram where we denote the numbers d_X as follows:

$$\underline{0} \text{ --- } \textcolor{red}{/} \underline{\bar{R}_{M-1}} \textcolor{red}{/} \underline{\bar{R}_{M-2}} \textcolor{red}{/} \cdots \textcolor{red}{/} \underline{\bar{R}_1} \textcolor{red}{/} \underline{n} \textcolor{blue}{\backslash} \underline{\bar{C}_1} \textcolor{blue}{\backslash} \cdots \textcolor{blue}{\backslash} \underline{\bar{C}_{N-2}} \textcolor{blue}{\backslash} \underline{\bar{C}_{N-1}} \textcolor{blue}{\backslash} \underline{0}$$

We have

$$\bar{R}_i = \sum_{l=M-i}^M r_l, \quad i = 1, \dots, M-1, \quad \bar{C}_j = \sum_{l=j+1}^N c_l, \quad j = 1, \dots, N-1.$$

Note that the assumption that \mathcal{D} is admissible and essential gives that all r_i and c_j are nonzero.

Definition 8.1. The *resolution* $\text{Res}(\mathcal{D})$ of \mathcal{D} is the brane diagram defined as

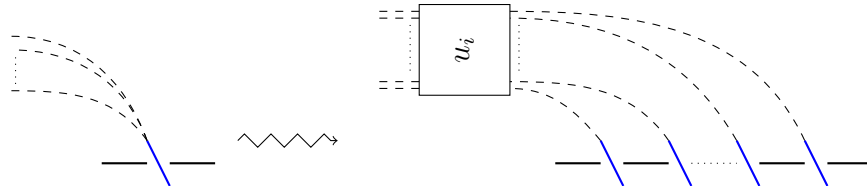
$$\underline{0} \text{ --- } \textcolor{red}{/} \underline{\bar{R}_{M-1}} \textcolor{red}{/} \underline{\bar{R}_{M-2}} \textcolor{red}{/} \cdots \textcolor{red}{/} \underline{\bar{R}_1} \textcolor{red}{/} \underline{n} \textcolor{blue}{\backslash} \underline{n-1} \textcolor{blue}{\backslash} \cdots \textcolor{blue}{\backslash} \underline{2} \textcolor{blue}{\backslash} \underline{1} \textcolor{blue}{\backslash} \underline{0}$$

That is, the resolved brane diagram $\text{Res}(\mathcal{D})$ is obtained from \mathcal{D} by replacing the part $n \textcolor{blue}{\backslash} \bar{C}_1 \textcolor{blue}{\backslash} \cdots \textcolor{blue}{\backslash} \bar{C}_{N-1} \textcolor{blue}{\backslash} 0$ with $n \textcolor{blue}{\backslash} n-1 \textcolor{blue}{\backslash} \cdots \textcolor{blue}{\backslash} 2 \textcolor{blue}{\backslash} 1 \textcolor{blue}{\backslash} 0$. Thus, with the notation from Subsection 3, $\text{Res}(\mathcal{D})$ is equal to the brane diagram $\mathcal{D}(R_1, \dots, R_{M-1}; n)$.

Given $u = u_1 \times \cdots \times u_N \in S_c$, we obtain an inclusion

$$\text{Res}_u: \text{Tie}(\mathcal{D}) \hookrightarrow \text{Tie}(\text{Res}(\mathcal{D})), \quad (8.1)$$

where for a tie diagram $D \in \text{Tie}(\mathcal{D})$ the resolved tie diagram $\text{Res}_u(D)$ is obtained via performing at each blue line U_i with ties the local move:



Here, the box around u_i represents an arbitrary diagram for u_i .

Example 8.2. Let D be the tie diagram:

$$\underline{0} \text{ --- } \textcolor{red}{/} \underline{1} \textcolor{red}{/} \underline{2} \textcolor{red}{/} \underline{3} \textcolor{red}{/} \underline{5} \textcolor{blue}{\backslash} \underline{3} \textcolor{blue}{\backslash} \underline{0} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We choose $u_1 = 21$ and $u_2 = 231$. Then, $\text{Res}_u(D)$ is given by

$$\underline{0} \text{ --- } \textcolor{red}{/} \underline{1} \textcolor{red}{/} \underline{2} \textcolor{red}{/} \underline{3} \textcolor{red}{/} \underline{5} \textcolor{blue}{\backslash} \underline{4} \textcolor{blue}{\backslash} \underline{3} \textcolor{blue}{\backslash} \underline{2} \textcolor{blue}{\backslash} \underline{1} \textcolor{blue}{\backslash} \underline{0} \quad \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Here, we see the rotated diagrams for u_1 and u_2 . The diagram for u_1 involves the ties of the first and the second blue line and the diagram for u_2 involves the ties of the third, fourth and fifth blue line.

For $D \in \text{Tie}(\mathcal{D})$, we set $\tilde{w}_D := \tilde{w}_{M(D)}$, where $\tilde{w}_{M(D)}$ is the permutation defined in Definition 7.7.

In terms of permutations, we can characterize $\text{Res}_u(D)$ as follows:

Proposition 8.3. *We have*

$$\text{Res}_u(D) = D_{u\tilde{w}_D S_r},$$

where $D_{u\tilde{w}_D S_r}$ is defined as in Subsection 3.5.

Proof. Suppose the blue line U_j in D is connected to the red lines $V_{i_1}, \dots, V_{i_{c_j}}$, where $i_1 < \dots < i_{c_j}$. Then, in $\text{Res}_u(D)$, the blue line $U_{C_{j-1}+l}$ is connected to $V_{i_{u(l)}}$ for $l = 1, \dots, c_j$. On the other hand, by construction of \tilde{w}_D , we have $\tilde{w}_D^{-1}(C_{j-1} + l) \in \{R_{i_{l-1}} + 1, \dots, R_{i_1}\}$ for all l . Thus, $\tilde{w}_D^{-1}u^{-1}(C_{j-1} + l) \in \{R_{i_{u_j(l)-1}} + 1, \dots, R_{i_{u_j(l)}}\}$ which proves the claim. \square

8.2. Equivariant resolution theorem. The equivariant resolution theorem connects the localization coefficients of stable basis elements of \mathcal{D} and $\text{Res}(\mathcal{D})$ as follows:

Theorem 8.4 (Equivariant resolution theorem). *Let D and D' be tie diagrams of \mathcal{D} . Then,*

$$\left(\prod_{i=1}^N \prod_{j=1}^{c_i-1} (jh)^{c_i-j} \right) \iota_D^*(\widetilde{\text{Stab}}_{\mathfrak{e}_-}(D')) = \Psi_{\mathcal{D}}(\iota_{\text{Res}_{u_0}(D)}^*(\text{Stab}_{\mathfrak{e}_-}(\text{Res}_u(D')))),$$

where $\widetilde{\text{Stab}}_{\mathfrak{e}_-}$ is the normalized version of $\text{Stab}_{\mathfrak{e}_-}$ from (5.3), $u \in S_{\mathfrak{e}}$, $u_0 = w_{0,c_1} \times \dots \times w_{0,c_N}$ and $w_{0,l}$ denotes the longest element in S_l for all l . The $\mathbb{Q}[h]$ -algebra homomorphism

$$\Psi_{\mathcal{D}}: \mathbb{Q}[t_1, \dots, t_n, h] \longrightarrow \mathbb{Q}[t_1, \dots, t_N, h]$$

is given by

$$\Psi_{\mathcal{D}}(t_{C_{i-1}+k}) = t_i - (k-1)h, \quad \text{for } i = 1, \dots, N, \ k = 1, \dots, c_i.$$

Applying Proposition 6.11 and Proposition 8.3 then directly implies the following:

Corollary 8.5. *With the notation of Theorem 8.4, we have*

$$\left(\prod_{i=1}^N \prod_{j=1}^{c_i-1} (jh)^{c_i-j} \right) \iota_D^*(\widetilde{\text{Stab}}_{\mathfrak{e}_-}(D')) = \Psi_{\mathcal{D}}(\iota_{w_D S_r}^*(\text{Stab}_{\mathfrak{e}_-}(w' S_r))),$$

where the stable basis element on the right is on $T^*F(R_1, \dots, R_{M-1}; n)$, $w' \in S_{\mathfrak{e}} \tilde{w}_{D'} S_r$ and $w_D = (w_{0,c_1} \times \dots \times w_{0,c_N}) \tilde{w}_D$.

Proof. By Proposition 8.3, $\text{Res}_{u_0}(D) = D_{w_D S_r}$ and $\text{Res}_u(D') = D_{w' S_r}$ for some $u' \in S_{\mathfrak{e}}$. Thus, Proposition 6.11 yields

$$\iota_{w_D S_r}^*(\text{Stab}_{\mathfrak{e}_-}(w' S_r)) = \iota_{\text{Res}_{u_0}(D)}^*(\text{Stab}_{\mathfrak{e}_-}(\text{Res}_u(D'))).$$

Hence, the claim follows from the equivariant resolution theorem. \square

Example 8.6. Given the tie diagrams

$$D = \begin{array}{ccccccccc} & & \text{---} & & \text{---} & & \text{---} & & \text{---} \\ & & \text{---} & & \text{---} & & \text{---} & & \text{---} \\ 0 & / & 1 & / & 3 & / & 5 & \backslash & 3 & \backslash & 2 & \backslash & 0 \\ & & \text{---} & & \text{---} & & \text{---} & & \text{---} & & \text{---} & & \text{---} \end{array} \quad D' = \begin{array}{ccccccccc} & & \text{---} & & \text{---} & & \text{---} & & \text{---} \\ & & \text{---} & & \text{---} & & \text{---} & & \text{---} \\ 0 & / & 1 & / & 3 & / & 5 & \backslash & 3 & \backslash & 2 & \backslash & 0 \\ & & \text{---} & & \text{---} & & \text{---} & & \text{---} & & \text{---} & & \text{---} \end{array}$$

In the following, we compute the localization coefficient $\iota_D^*(\widetilde{\text{Stab}}_{\mathfrak{c}_-}(D'))$. Note that $n = 5$, $\mathbf{r} = (2, 2, 1)$ and $\mathbf{c} = (2, 1, 2)$. Let $w, w' \in S_5$ be as in Example 6.10. Then, we have $Z(w) = M(D)$ and $Z(w') = M(D')$. Hence, we know from Example 6.10 that $\tilde{w}_D = 14253$ and $\tilde{w}_{D'} = 14235$. Thus, $w = (s \times \text{id} \times s)\tilde{w}_D$, where $s = 12 \in S_2$ which gives $w = w_D$. Therefore, by Corollary 8.5, we have

$$h^2 \iota_D^*(\widetilde{\text{Stab}}_{\mathfrak{c}_-}(D')) = \Psi_{\mathcal{D}}(\iota_{w_{S_{\mathbf{r}}}}^*(\text{Stab}_{\mathfrak{c}_-}(w'_{S_{\mathbf{r}}})) \quad (8.2)$$

where

$$\Psi_{\mathcal{D}}: \mathbb{Q}[t_1, t_2, t_3, t_4, t_5, h] \longrightarrow \mathbb{Q}[t_1, t_2, t_3, h]$$

is the $\mathbb{Q}[h]$ -algebra homomorphism given by $t_1 \mapsto t_1$, $t_2 \mapsto t_1 - h$, $t_3 \mapsto t_2$, $t_4 \mapsto t_3$, $t_5 \mapsto t_3 - h$. From (6.8), we know

$$\iota_{w_{S_{\mathbf{r}}}}^*(\text{Stab}_{\mathfrak{c}_-}(w'_{S_{\mathbf{r}}})) = h(t_1 - t_3 + h)(t_2 - t_3 + h)(t_2 - t_4 + h)(t_4 - t_5)(t_1 - t_2)(t_3 - t_5)(t_1 - t_5),$$

which yields

$$\Psi_{\mathcal{D}}(\iota_{w_{S_{\mathbf{r}}}}^*(\text{Stab}_{\mathfrak{c}_-}(w'_{S_{\mathbf{r}}})) = h^3(t_1 - t_2 + h)(t_1 - t_2)(t_1 - t_2 + h)(t_2 - t_3)(t_1 - t_3). \quad (8.3)$$

Finally, inserting (8.3) in (8.2) gives

$$\iota_D^*(\widetilde{\text{Stab}}_{\mathfrak{c}_-}(D')) = h(t_1 - t_2 + h)(t_1 - t_2)(t_1 - t_2 + h)(t_2 - t_3)(t_1 - t_3).$$

Remark 8.7. The equivariant resolution theorem as stated in [BR23, Theorem 6.13] connects the stable basis elements of different bow varieties which differ by resolving just one blue line. As explained in e.g. [Weh24, Section 9.6], Theorem 8.4 can be deduced from [BR23, Theorem 6.13] by an inductive procedure.

8.3. Approximations of localization coefficients. Next, we combine the diagrammatic localization formula and Corollary 8.5 to approximate localization coefficients of stable basis elements modulo powers of h .

For this, we like to choose the reduced diagrams for permutations of a particular form: let $w = u_0 \circ \tilde{w}_D \circ (v_1 \times \cdots \times v_M)$, where, as in Theorem 8.4, $u_0 = w_{0,c_1} \times \cdots \times w_{0,c_N}$ and $v_j \in S_{r_j}$ is an arbitrary element for $j = 1, \dots, M$. By Corollary 7.13, we can choose a reduced diagram for w of the form:

(8.4)

Here, the boxes represent reduced diagrams of the respective permutations. The example on the right shows the permutation $u_0 \circ \tilde{w}_D \circ v$, where \tilde{w}_D is the shortest $(S_{\mathbf{c}}, S_{\mathbf{r}})$ -double coset

representative from Example 7.6 for $\mathbf{r} = (3, 2, 2, 3)$, $\mathbf{c} = (2, 3, 2, 1, 2)$ and $v = v_1 \times v_2 \times v_3 \times v_4$, where $v_1 = 312$, $v_2 = 12$, $v_3 = 21$, $v_4 = 231$.

If d_w is a diagram of shape (8.4) then, according to their position in the diagram, we define the following subsets of crossings in d_w :

$$\begin{aligned} K_U(d_w) &= \{\kappa \in K(d_w) \mid \kappa \text{ belongs to some } w_{0,c_i} \text{ for } i = 1, \dots, N\}, \\ K_W(d_w) &= \{\kappa \in K(d_w) \mid \kappa \text{ belongs to } \tilde{w}_D\}, \\ K_V(d_w) &= \{\kappa \in K(d_w) \mid \kappa \text{ belongs to some } v_i \text{ for } i = 1, \dots, M\}. \end{aligned}$$

The next proposition shows that the weights of crossings in $K_U(d_w)$ precisely contribute the normalization factor which appears in Corollary 8.5.

Proposition 8.8. *We have*

$$\Psi_{\mathcal{D}}\left(\prod_{\kappa \in K_U(d_w)} \text{wt}(\kappa)\right) = \prod_{i=1}^N \prod_{j=1}^{c_i-1} (jh)^{c_i-j}.$$

The proof is immediate from the following lemma:

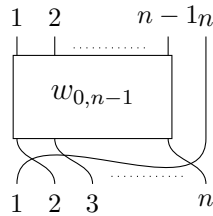
Lemma 8.9. *Let $w_{0,n} \in S_n$ be the longest element and $d_{w_{0,n}}$ a reduced diagram of $w_{0,n}$. Then,*

$$\Psi\left(\prod_{\kappa \in K(d_{w_{0,n}})} \text{wt}(\kappa)\right) = \prod_{j=1}^{n-1} (jh)^{n-j},$$

where $\Psi: \mathbb{Q}[t_1, \dots, t_n, h] \rightarrow \mathbb{Q}[t, h]$ is the $\mathbb{Q}[h]$ -algebra homomorphism given by

$$t_i \mapsto t - (i-1)h, \quad i = 1, \dots, n.$$

Proof. By (6.4), the product $\prod_{\kappa \in K(d_{w_{0,n}})} \text{wt}(\kappa)$ does not depend on the choice of reduced diagram. We prove the statement by induction on n where the case $n = 1$ is clear. For $n > 1$, we choose $d_{w_{0,n}}$ to be of the following shape:



Here, the box represents a reduced diagram for $w_{0,n-1}$. Let K' be the set of crossings contained in the box of $w_{0,n-1}$ and K'' be the set of crossings outside of the box of $w_{0,n-1}$. From the diagram $d_{w_{0,n}}$, we can read off that the crossings in K'' have weights $t_1 - t_n, \dots, t_{n-1} - t_n$.

Thus, we have $\Psi(\prod_{\kappa \in K''} \text{wt}(\kappa)) = \prod_{i=1}^n (ih)$. Applying the induction hypothesis to K' yields

$$\begin{aligned} \Psi\left(\prod_{\kappa \in K(d_{w_0, n})} \text{wt}(\kappa)\right) &= \Psi\left(\prod_{\kappa \in K'} \text{wt}(\kappa)\right) \cdot \Psi\left(\prod_{\kappa \in K''} \text{wt}(\kappa)\right) \\ &= \left(\prod_{j=1}^{n-2} (jh)^{n-1-j}\right) \cdot \Psi\left(\prod_{i=1}^{n-1} (t_i - t_n)\right) \\ &= \prod_{j=1}^{n-1} (jh)^{n-j} \end{aligned}$$

which finishes the proof. \square

Proposition 8.10 (Approximation). *Under the assumptions as in Corollary 8.5, we have*

$$\iota_D^*(\widetilde{\text{Stab}}_{\mathfrak{C}_-}(D')) \equiv \sum_{z \in w_D S_{\mathbf{r}}} \frac{(-1)^{\ell(w') + \ell(w' S_{\mathbf{r}})} \left(\prod_{\alpha \in L'_z} \Psi_{\mathcal{D}}(\alpha + h) \right) \cdot P_{d_z, w', m}}{\prod_{\beta \in R_{\mathbf{r}}} \Psi_{\mathcal{D}}(z \cdot \beta)} \pmod{h^m}, \quad (8.5)$$

where L'_z is defined as in Proposition 6.7,

$$P_{d_z, w', m} = \sum_{K' \in K(d_z, w', m-1)} \left(h^{|K' \setminus K_U(d_z)|} f_{K'} \cdot \left(\prod_{\substack{\kappa \in K(d_z) \\ \kappa \notin K' \cup K_U(d_z)}} \Psi_{\mathcal{D}}(\text{wt}(\kappa)) \right) \right)$$

and

$$\begin{aligned} K(d_z, w', m-1) &= \{K' \in K_{d_z, w'} \mid |K' \setminus K_U(d_z)| \leq m-1\}, \\ f_{K'} &= \frac{h^{|K' \cap K_U(d_z)|} \prod_{\kappa \in K_U(d_z) \setminus K'} \Psi_{\mathcal{D}}(\text{wt}(\kappa))}{\prod_{i=1}^N \prod_{j=1}^{c_i-1} (jh)^{c_i-j}}. \end{aligned}$$

Remark 8.11. By Proposition 8.8, the factor $f_{K'}$ is always contained in \mathbb{Q} . Moreover, as $F_z(i) \neq F_z(j)$ for all $t_i - t_j \in R_{\mathbf{r}}$ and $z \in S_{\mathfrak{C}} \tilde{w}_D S_{\mathbf{r}}$, we deduce that for all $\alpha \in R_{\mathbf{r}}$, the polynomial $\Psi_{\mathcal{D}}(z \cdot \alpha)$ is contained in S , as defined in (4.1).

Proof of Proposition 8.10. For $z \in w_D S_{\mathbf{r}}$ with reduced diagram d_z of shape (8.4) let $m_0(z) = |K(d_z) \setminus K_U(d_z)|$. By Corollary 8.5, we have

$$\iota_D^*(\widetilde{\text{Stab}}_{\mathfrak{C}_-}(D')) = \sum_{z \in w_D S_{\mathbf{r}}} \frac{(-1)^{\ell(w') + \ell(w' S_{\mathbf{r}})} \left(\prod_{\alpha \in L'_z} \Psi_{\mathcal{D}}(\alpha + h) \right) \cdot P_{d_z, w', m_0(z)}}{\prod_{\beta \in R_{\mathbf{r}}} \Psi_{\mathcal{D}}(z \cdot \beta)}.$$

If $K' \in K_{d_z, w'} \setminus K(d_z, w', m-1)$ then by Proposition 8.8, $h^{|K'|} \prod_{\kappa \in K_U(d_z) \setminus K'} \Psi_{\mathcal{D}}(\text{wt}(\kappa))$ is divisible by $h^{\frac{1}{2}(c_1(c_1-1) + \dots + c_N(c_N-1)) + m}$. Hence, the contribution of those K' in $P_{d_z, w', m_0(z)}$ vanishes modulo h^m . Thus, we have

$$P_{d_z, w', m_0(z)} \equiv P_{d_z, w', m} \pmod{h^m}, \quad \text{for all } z \in w_D S_{\mathbf{r}}$$

which proves the proposition. \square

9. CHEVALLEY–MONK FORMULAS IN THE SEPARATED CASE

In this section, we state and prove Chevalley–Monk formulas for the tautological bundles of bow varieties corresponding to separated brane diagrams. In the upcoming section, we then derive Chevalley–Monk formulas for tautological bundles corresponding to general bow varieties using Hanany–Witten transition.

Assumption. For this section, we assume that \mathcal{D} is a separated brane diagram.

Recall from Subsection 2.8 that the tautological bundles $\xi_{M+1}, \dots, \xi_{M+N+1}$ are trivial. Hence, we focus on characterizing the multiplication of $c_1(\xi_1), \dots, c_1(\xi_M)$.

9.1. Chevalley–Monk formula for the antidominant chamber. We first restrict our attention to the antidominant chamber \mathfrak{C}_- . In this case, the Chevalley–Monk formula is given as follows:

Theorem 9.1 (Chevalley–Monk for antidominant chamber). *Let $D \in \text{Tie}(\mathcal{D})$. Then, we have the following identity in $H_{\mathbb{T}}^*(\mathcal{C}(\mathcal{D}))_{\text{loc}}$:*

$$c_1(\xi_i) \cup \text{Stab}_{\mathfrak{C}_-}(D) = \iota_D^*(c_1(\xi_i)) \cdot \text{Stab}_{\mathfrak{C}_-}(D) + \sum_{D' \in \text{SM}_{D,i}} \text{sgn}(D, D') h \cdot \text{Stab}_{\mathfrak{C}_-}(D'),$$

for $i = 1, \dots, M$. Here, the set of simple moves $\text{SM}_{D,i}$ is defined in (9.1) and the signs of simple moves $\text{sgn}(D, D') \in \{\pm 1\}$ are defined in Definition 9.4.

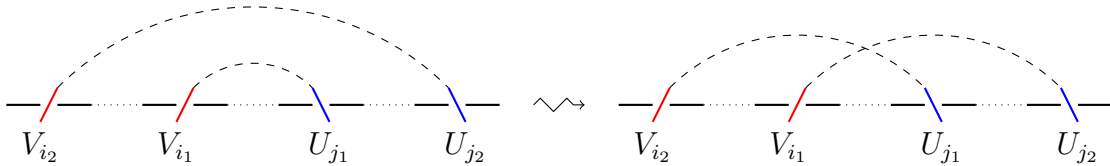
The proof of Theorem 9.1 is given in Subsection 9.4. We first give the definitions relevant for the theorem. We begin with introducing notion of moving ties:

Definition 9.2. Let $D, D' \in \text{Tie}(\mathcal{D})$. We say that D' is obtained from D via a *simple move* if there exist $1 \leq i_1 < i_2 \leq M$ and $1 \leq j_1 < j_2 \leq N$ such that $(V_{i_1}, U_{j_1}), (V_{i_2}, U_{j_2}) \in D$, $(V_{i_1}, U_{j_2}), (V_{i_2}, U_{j_1}) \in D'$ and

$$D \setminus \{(V_{i_1}, U_{j_2}), (V_{i_2}, U_{j_1})\} = D' \setminus \{(V_{i_1}, U_{j_1}), (V_{i_2}, U_{j_2})\}.$$

We call (V_{i_1}, U_{j_1}) the *right moving tie* and (V_{i_2}, U_{j_2}) the *left moving tie* of D . Let SM_D be the set of all tie diagrams that are obtained from D via a simple move.

Pictorially, simple moves can be described as switching two ties as illustrated:



Translating between tie diagrams and binary contingency tables immediately gives the following equivalent characterization of simple moves:

Lemma 9.3. *Let $D, D' \in \text{Tie}(\mathcal{D})$. Then, D' is obtained from D via a simple move if and only if there exist $1 \leq i_1 < i_2 \leq M$ and $1 \leq j_1 < j_2 \leq N$ such that*

- (i) $M(D)_{i_1, j_1} = M(D)_{i_2, j_2} = 1$ and $M(D)_{i_1, j_2} = M(D)_{i_2, j_1} = 0$,
- (ii) $M(D')_{i_1, j_1} = M(D')_{i_2, j_2} = 0$ and $M(D')_{i_1, j_2} = M(D')_{i_2, j_1} = 1$,
- (iii) $M(D)_{l, k} = M(D')_{l, k}$, for $(l, k) \notin \{(i_1, j_1), (i_2, j_1), (i_1, j_2), (i_2, j_2)\}$.

For $X_i \in h(\mathcal{D})$ with $i \in \{1, \dots, M\}$, we define the *set of simple moves relative to X_i* as

$$\text{SM}_{D,i} = \{D' \in \text{Tie}(\mathcal{D}) \mid D' \text{ satisfies (a), (b) and (c)}\}, \quad (9.1)$$

where

- (a) D' is obtained from D via a simple move,
- (b) if (V_{i_1}, U_{j_1}) is the right moving tie of D then $X_i \triangleleft V_{i_1}$,
- (c) if (V_{i_2}, U_{j_2}) is the left moving tie of D then $V_{i_2} \triangleleft X_i$.

Next, we define the sign of a simple move:

Definition 9.4. Given $D' \in \text{SM}_D$ with right moving tie (V_{i_1}, U_{j_1}) and left moving tie (V_{i_2}, U_{j_2}) . Then, we define

$$\text{sgn}(D, D') := \begin{cases} 1 & \text{if } n_1 + n_2 \text{ is even,} \\ -1 & \text{if } n_1 + n_2 \text{ is odd,} \end{cases}$$

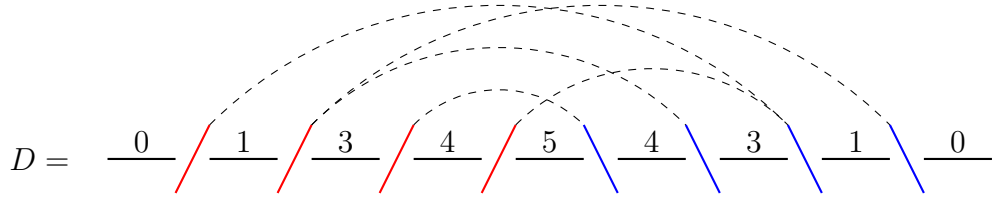
where

$$n_1 := |\{(V_{i_1}, U_j) \in D \mid j_1 < j < j_2\}|, \quad n_2 := |\{(V_{i_2}, U_j) \in D \mid j_1 < j < j_2\}|.$$

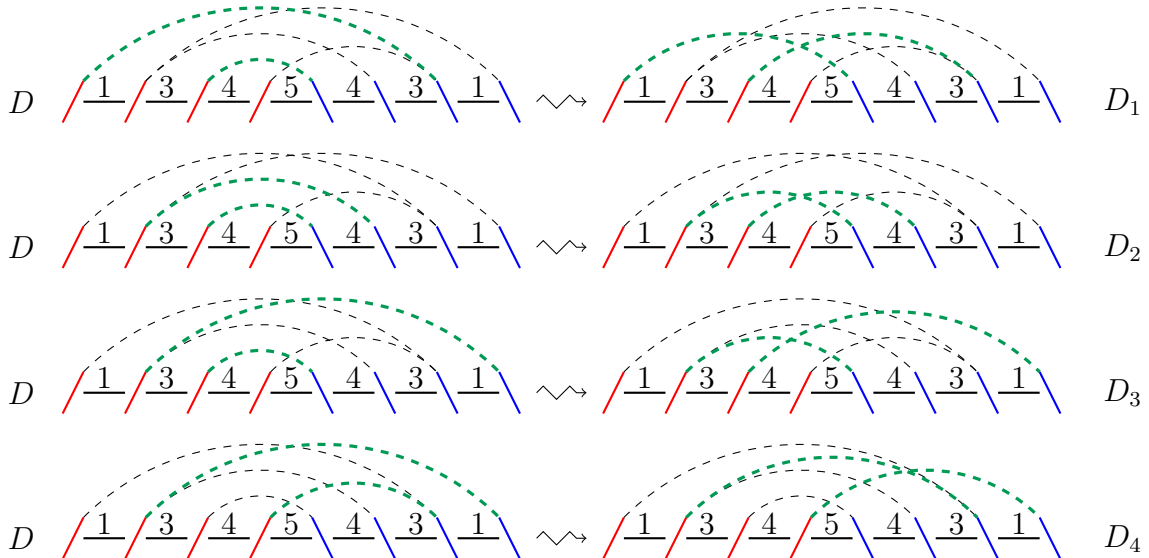
We call $\text{sgn}(D, D')$ the *sign of the simple move between D and D'* .

Thus, all notions appearing in Theorem 9.1 are introduced.

Example 9.5. Let $\mathcal{D} = 0/1/3/4/5/4/3/1/0$ and



In the following, we determine $c_1(\xi_i) \cup \text{Stab}_{\mathfrak{C}_-}(D)$ for $i = 3$. The simple moves that are contained in $\text{SM}_{D,i}$ are illustrated as follows:



That is $\text{SM}_{D,i} = \{D_1, D_2, D_3, D_4\}$, where

$$\begin{aligned} D_1 &= (D \cup \{(V_2, U_3), (V_4, U_1)\}) \setminus \{(V_2, U_1), (V_4, U_3)\}, \\ D_2 &= (D \cup \{(V_2, U_2), (V_3, U_1)\}) \setminus \{(V_2, U_1), (V_3, U_2)\}, \\ D_3 &= (D \cup \{(V_2, U_4), (V_3, U_1)\}) \setminus \{(V_2, U_1), (V_3, U_4)\}, \\ D_4 &= (D \cup \{(V_1, U_4), (V_3, U_3)\}) \setminus \{(V_1, U_3), (V_3, U_4)\}. \end{aligned} \quad (9.2)$$

Note that in the illustration of the simple moves, we omitted the horizontal black lines on the boundary of the tie diagrams. From the diagrams one can easily read off the respective signs:

$$\text{sgn}(D, D_1) = \text{sgn}(D, D_2) = \text{sgn}(D, D_4) = 1, \quad \text{sgn}(D, D_3) = -1.$$

By (2.6), we have an isomorphism of \mathbb{T} -representations $\iota_D^*(\xi_i) \cong \mathbb{C}_{U_2} \oplus \mathbb{C}_{U_3} \oplus \mathbb{C}_{U_4}$. Thus, $\iota_D^*(c_1(\xi_i)) = t_2 + t_3 + t_4$. Hence, Theorem 9.1 gives

$$\begin{aligned} c_1(\xi_i) \cup \text{Stab}_{\mathfrak{C}_-}(D) &= (t_2 + t_3 + t_4)\text{Stab}_{\mathfrak{C}_-}(D) + h\text{Stab}_{\mathfrak{C}_-}(D_1) + h\text{Stab}_{\mathfrak{C}_-}(D_2) \\ &\quad - h\text{Stab}_{\mathfrak{C}_-}(D_3) + h\text{Stab}_{\mathfrak{C}_-}(D_4). \end{aligned}$$

Remark 9.6. The notion of simple moves on brane diagrams also appeared in [FS23], where they are called *swap moves*. There these moves were used to classify torus invariant curves on bow varieties.

9.2. The sign. In this subsection, we give an interpretation of $\text{sgn}(D, D')$ in terms of permutations assigned to the double cosets of D and D' . From this, we deduce that after appropriate normalization of the stable basis all off-diagonal entries in the Chevalley–Monk formula become equal to $-h$.

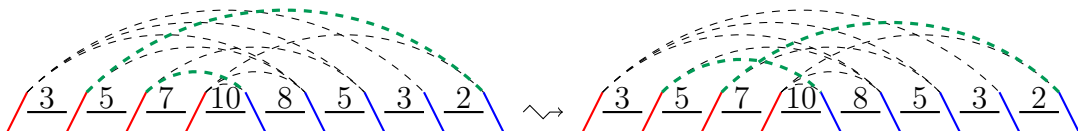
Given a tie diagram D and $D' \in \text{SM}_D$ with left moving tie (V_{i_1}, U_{j_1}) and right moving tie (V_{i_2}, U_{j_2}) . Let $\tilde{w}_{D'} = \tilde{w}_{M(D')} \in S_n$ be the shortest $(S_{\mathfrak{C}}, S_{\mathfrak{C}})$ -double coset representative from Definition 7.7. By definition, there exist a unique $f_1 \in \{R_{i_1-1} + 1, \dots, R_{i_1}\}$ such that $\tilde{w}_{D'}(f_1) \in \{C_{j_2-1} + 1, \dots, C_{j_2}\}$ and a unique $f_2 \in \{R_{i_2-1} + 1, \dots, R_{i_2}\}$ with $\tilde{w}_{D'}(f_2) \in \{C_{j_1-1} + 1, \dots, C_{j_1}\}$. We set

$$\tilde{y}_D := \tilde{w}_{D'} \circ (f_1, f_2). \quad (9.3)$$

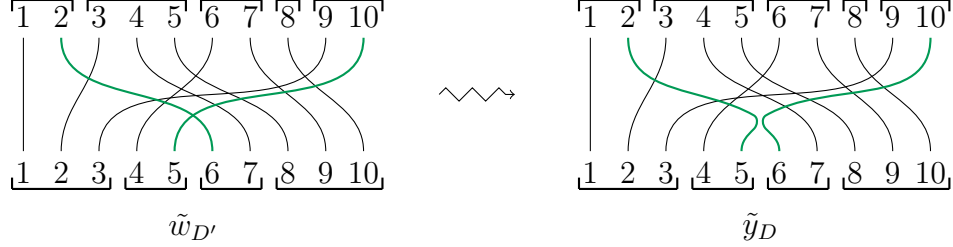
Then, by construction, $\tilde{y}_D \in \tilde{w}_D S_{\mathbf{r}}$.

The permutation \tilde{y}_D has the following diagrammatic interpretation: let $d_{\tilde{w}_{D'}}$ be a reduced diagram of $\tilde{w}_{D'}$. Since $(V_{i_1}, U_{j_2}) \in D'$ there exists a unique strand λ_1 in $d_{\tilde{w}_{D'}}$ starting in $\{R_{i_1-1} + 1, \dots, R_{i_1}\}$ and ending in $\{C_{j_2-1} + 1, \dots, C_{j_2}\}$. Likewise, as $(V_{i_2}, U_{j_1}) \in D'$ there is also a unique strand λ_2 in $d_{\tilde{w}_{D'}}$ which starts in $\{R_{i_2-1} + 1, \dots, R_{i_2}\}$ and ends in $\{C_{j_1-1} + 1, \dots, C_{j_1}\}$. As $i_1 < i_2$ and $j_1 < j_2$, the strands λ_1 and λ_2 intersect exactly once. Resolving the crossing of λ_1 and λ_2 then gives a diagram for the permutation \tilde{y}_D .

Example 9.7. Consider the simple move:



We denote the tie diagram on the left by D and the one on the right by D' . The moving ties are (V_2, U_1) and (V_3, U_5) . Note that $n = 10$, $\mathbf{r} = (3, 2, 2, 3)$ and $\mathbf{c} = (2, 3, 2, 1, 2)$. By construction, the binary contingency table $M(D')$ equals the matrix A from Example 7.6 where we also constructed the corresponding shortest $(S_{\mathbf{c}}, S_{\mathbf{r}})$ -double coset representative $\tilde{w}_{D'} = 13961024578$. We have $\tilde{w}_{D'}(5) \in \{9, 10\} = \{C_3 + 1, \dots, C_4\}$ and $\tilde{w}_{D'}(6) \in \{1, 2\} = \{C_0 + 1, \dots, C_1\}$, so $f_1 = 6$ and $f_2 = 7$. This gives $\tilde{y}_D = 13962104578$. The diagrammatic construction of \tilde{y}_D is illustrated as follows:



Here, we highlighted the strands whose resolution gives \tilde{y}_D .

Comparing the length of \tilde{w}_D and \tilde{y}_D gives the sign we attached to D and D' :

Proposition 9.8. *We have $(-1)^{\ell(\tilde{w}_D) + \ell(\tilde{y}_D)} = \text{sgn}(D, D')$.*

Proof. By construction, $\tilde{w}_{D'}(R_{i-1} + l) \in \{C_{F_{M(D'),i}(l)-1} + 1, \dots, C_{F_{M(D'),i}(l)}\}$ for all i, l , where we used the notation from Subsection 7.2. This directly implies that the set

$$\{(i, j) \mid \text{there exists } l \text{ with } R_{l-1} + 1 \leq i < j \leq R_l \text{ and } \tilde{y}_D(i) > \tilde{y}_D(j)\}$$

equals

$$\{(f_1, f_1 + 1), \dots, (f_1, f_1 + n_1)\} \cup \{(f_2 - 1, f_2), \dots, (f_2 - n_2, f_2)\}.$$

Here, n_1 and n_2 are defined as in Definition 9.4. Since \tilde{w}_D is the shortest representative of $\tilde{y}_D S_{\mathbf{r}}$, we conclude $\ell(\tilde{y}_D) = \ell(\tilde{w}_D) + n_1 + n_2$ which proves the proposition. \square

Theorem 9.1 implies now a Chevalley–Monk formula with simplified signs:

Corollary 9.9. *Let $D \in \text{Tie}(\mathcal{D})$. Then, the following identity holds in $H_{\mathbb{T}}^*(\mathcal{C}(\mathcal{D}))_{\text{loc}}$:*

$$c_1(\xi_i) \cup \text{Stab}'_{\mathfrak{C}_-}(D) = \iota_D^*(c_1(\xi_i)) \cdot \text{Stab}'_{\mathfrak{C}_-}(D) - h \cdot \left(\sum_{D' \in \text{SM}_{D,i}} \text{Stab}'_{\mathfrak{C}_-}(D') \right),$$

for $i = 1, \dots, M$, where $\text{Stab}'_{\mathfrak{C}_-}(T) = (-1)^{\ell(\tilde{w}_T)} \text{Stab}_{\mathfrak{C}_-}(T)$ for $T \in \text{Tie}(\mathcal{D})$.

Proof. By construction, \tilde{y}_D is obtained from $\tilde{w}_{D'}$ by precomposition with a transposition. Hence, we have $(-1)^{\ell(\tilde{y}_D)} = (-1)^{\ell(\tilde{w}_{D'})+1}$. Thus, Proposition 9.8 yields

$$\text{sgn}(D, D') = (-1)^{\ell(\tilde{w}_D) + \ell(\tilde{y}_D)} = (-1)^{\ell(\tilde{w}_D) + \ell(\tilde{w}_{D'}) + 1}$$

which proves the corollary. \square

9.3. Divisibility and Approximation. We now consider divisibility and approximation results for localization coefficients of stable basis elements. These results are essential ingredients of the proof of Theorem 9.1. We first formulate the results and deduce some consequences. The proofs are then given in Subsections 9.5 and 9.6.

Proposition 9.10 (h^2 -Divisibility). *The localization coefficient $\iota_{D'}^*(\text{Stab}_{\mathfrak{C}_-}(D))$ is divisible by h^2 , for $D \in \text{Tie}(\mathcal{D})$ and $D' \notin \text{SM}_D \cup \{D\}$.*

We prove Proposition 9.10 in Subsection 9.5. By applying Theorem 5.3, we deduce the analogous result for the chamber \mathfrak{C}_+ :

Corollary 9.11. *We have that $\iota_D^*(\text{Stab}_{\mathfrak{C}_+}(D'))$ is divisible by h^2 for $D \in \text{Tie}(\mathcal{D})$ and $D' \notin \text{SM}_D \cup \{D\}$.*

Proof. As before, let $w_{0,N} \in S_N$ be the longest element. By Theorem 5.3, we have

$$\iota_D^*(\widetilde{\text{Stab}}_{\mathfrak{C}_+}(D')) = w_{0,N} \cdot (\iota_{w_{0,N}.D}^*(\widetilde{\text{Stab}}_{\mathfrak{C}_-}(w_{0,N}.D')).) \quad (9.4)$$

Since $D' \notin \text{SM}_D$ if and only if $w_{0,N}.D \notin \text{SM}_{w_{0,N}.D'}$, Proposition 9.10 implies that the right hand side of (9.4) is divisible by h^2 . Thus, $\iota_D^*(\widetilde{\text{Stab}}_{\mathfrak{C}_+}(D'))$ is divisible by h^2 and hence also $\iota_D^*(\text{Stab}_{\mathfrak{C}_+}(D'))$. \square

Combining Proposition 9.10 and Corollary 9.11 gives the following divisibility :

Corollary 9.12 (h^2 -Divisibility of products). *Let $D, D', T \in \text{Tie}(\mathcal{D})$ such that $T \notin \{D, D'\}$ or $D' \notin \text{SM}_D \cup \{D\}$. Then, we have*

$$\iota_T^*(\text{Stab}_{\mathfrak{C}_-}(D) \cup \text{Stab}_{\mathfrak{C}_+}(D')) \equiv 0 \pmod{h^2}. \quad (9.5)$$

Proof. If $T \notin \{D, D'\}$ then the smallness condition implies that both $\iota_T^*(\text{Stab}_{\mathfrak{C}_-}(D))$ and $\iota_T^*(\text{Stab}_{\mathfrak{C}_+}(D'))$ are divisible by h which gives that (9.5) is divisible by h^2 . If $T = D$ and $D' \notin \text{SM}_D \cup \{D\}$ then, by Corollary 9.11, $\iota_T^*(\text{Stab}_{\mathfrak{C}_+}(D'))$ is divisible by h^2 and so is (9.5). Likewise, if $T = D'$ and $D' \notin \text{SM}_D \cup \{D\}$ then Proposition 9.10 implies that $\iota_T^*(\text{Stab}_{\mathfrak{C}_-}(D))$ is divisible by h^2 and hence also (9.5). \square

We proceed with the following statement about h^2 -approximations of localization coefficients:

Proposition 9.13 (h^2 -Approximation). *Let $D \in \text{Tie}(\mathcal{D})$ and $D' \in \text{SM}_D$ with (V_{i_1}, U_{j_1}) be the right moving tie and (V_{i_2}, U_{j_2}) be the left moving tie of D . Then, we have*

$$\frac{\iota_{D'}^*(\text{Stab}_{\mathfrak{C}_-}(D))}{e_{\mathbb{T}}(T_D \mathcal{C}(\mathcal{D})_{\mathfrak{C}_-})} \equiv \text{sgn}(D, D') \frac{h}{t_{j_1} - t_{j_2}} \pmod{h^2}$$

in $S^{-1}H_{\mathbb{T}}^*(\mathcal{C}(\mathcal{D}))$. Here, S is defined as in (4.1).

As before, Theorem 5.3 gives the analogous statement for \mathfrak{C}_+ :

Corollary 9.14. *Let $D \in \text{Tie}(\mathcal{D})$ and $D' \in \text{SM}_D$ with (V_{i_1}, U_{j_1}) be the right moving tie and (V_{i_2}, U_{j_2}) be the left moving tie of D . Then, we have*

$$\frac{\iota_{D'}^*(\text{Stab}_{\mathfrak{C}_+}(D'))}{e_{\mathbb{T}}(T_D \mathcal{C}(\mathcal{D})_{\mathfrak{C}_+})} \equiv \text{sgn}(D, D') \frac{h}{t_{j_2} - t_{j_1}} \pmod{h^2}$$

in $S^{-1}H_{\mathbb{T}}^*(\mathcal{C}(\mathcal{D}))$.

Proof. By the definition of $\widetilde{\text{Stab}}$, see (5.3), we have

$$\frac{\iota_D^*(\text{Stab}_{\mathfrak{C}_+}(D'))}{e_{\mathbb{T}}(T_D\mathcal{C}(\mathcal{D})_{\mathfrak{C}_+}^-)} = \frac{\iota_D^*(\widetilde{\text{Stab}}_{\mathfrak{C}_+}(D'))}{\iota_D^*(\widetilde{\text{Stab}}_{\mathfrak{C}_+}(D))}, \quad \frac{\iota_D^*(\text{Stab}_{\mathfrak{C}_-}(D'))}{e_{\mathbb{T}}(T_D\mathcal{C}(\mathcal{D})_{\mathfrak{C}_-}^-)} = \frac{\iota_D^*(\widetilde{\text{Stab}}_{\mathfrak{C}_-}(D'))}{\iota_D^*(\widetilde{\text{Stab}}_{\mathfrak{C}_-}(D))}. \quad (9.6)$$

In addition, Theorem 5.3 yields

$$\frac{\iota_D^*(\widetilde{\text{Stab}}_{\mathfrak{C}_+}(D'))}{\iota_D^*(\widetilde{\text{Stab}}_{\mathfrak{C}_+}(D))} = w_{0,N} \cdot \left(\frac{\iota_{w_{0,N}.D}^*(\widetilde{\text{Stab}}_{\mathfrak{C}_-}(w_{0,N}.D'))}{\iota_{w_{0,N}.D}^*(\widetilde{\text{Stab}}_{\mathfrak{C}_-}(w_{0,N}.D))} \right). \quad (9.7)$$

The tie diagram $w_{0,N}.D$ is obtained from $w_{0,N}.D'$ via a simple move where (V_{i_1}, U_{N-j_2+1}) is the right moving tie and (V_{i_2}, U_{N-j_1+1}) is the left moving tie of $w_{0,N}.D'$. Thus, we have

$$\begin{aligned} \frac{\iota_D^*(\text{Stab}_{\mathfrak{C}_+}(D'))}{e_{\mathbb{T}}(T_D\mathcal{C}(\mathcal{D})_{\mathfrak{C}_+}^-)} &= w_{0,N} \cdot \left(\frac{\iota_{w_{0,N}.D}^*(\text{Stab}_{\mathfrak{C}_-}(w_{0,N}.D'))}{e_{\mathbb{T}}(T_{w_{0,N}.D}\mathcal{C}(\mathcal{D})_{\mathfrak{C}_-}^-)} \right) \\ &\equiv w_{0,N} \cdot \left(\text{sgn}(w_{0,N}.D', w_{0,N}.D) \frac{h}{t_{N-j_2+1} - t_{N-j_1+1}} \right) \pmod{h^2} \\ &\equiv \text{sgn}(D, D') \frac{h}{t_{j_2} - t_{j_1}} \pmod{h^2}, \end{aligned}$$

where the first equality follows from (9.6) and (9.7), the subsequent congruence from Proposition 9.13 and the final congruence from $\text{sgn}(D, D') = \text{sgn}(w_{0,N}.D', w_{0,N}.D)$. \square

Remark 9.15. In the framework of partial flag varieties, the results of this subsection are contained in [Su17a, Corollary 3.11].

9.4. Proof of Theorem 9.1. We begin with the following auxiliary statement:

Lemma 9.16. *Let $D \in \text{Tie}(\mathcal{D})$, $D' \in \text{SM}_D$ with right moving tie (V_{i_1}, U_{j_1}) and left moving tie (V_{i_2}, U_{j_2}) . Then, we have*

$$\iota_{D'}^*(c_1(\xi_i)) - \iota_D^*(c_1(\xi_i)) \equiv \begin{cases} t_{j_1} - t_{j_2} \pmod{h} & \text{if } D \in \text{SM}_{D,i}, \\ 0 \pmod{h} & \text{if } D \notin \text{SM}_{D,i}. \end{cases}$$

Proof. From (2.6), we immediately obtain

$$\iota_T^*(c_1(\xi_i)) = \sum_{U \in \mathfrak{b}(\mathcal{D})} d_{T,U,X_i} t_i \pmod{h}, \quad \text{for all } T \in \text{Tie}(\mathcal{D}), \quad (9.8)$$

where d_{T,U,X_i} is defined as in Subsection 2.8. According to the relative position of X_i with respect to V_{i_1} and V_{i_2} we have the three cases illustrated in Figure 1. By construction, in the first and third case holds $d_{D,U,X_i} = d_{D',U,X_i}$ for all $U \in \mathfrak{b}(\mathcal{D})$. Hence, by (9.8), we have $\iota_{D'}^*(c_1(\xi_i)) - \iota_D^*(c_1(\xi_i)) \equiv 0 \pmod{h}$ for $D' \notin \text{SM}_{D,i}$. The second case is equivalent to $D' \in \text{SM}_{D,i}$. In this case, we have

$$d_{D,U,X_i} = \begin{cases} d_{D',U,X_i} & \text{if } U \in \mathfrak{b}(\mathcal{D}) \setminus \{U_{j_1}, U_{j_2}\}, \\ d_{D',U,X_i} - 1 & \text{if } U = U_{j_1}, \\ d_{D',U,X_i} + 1 & \text{if } U = U_{j_2}. \end{cases}$$

Thus, (9.8) gives $\iota_{D'}^*(c_1(\xi_i)) - \iota_D^*(c_1(\xi_i)) \equiv t_{j_1} - t_{j_2} \pmod{h}$ for $D' \in \text{SM}_{D,i}$. \square

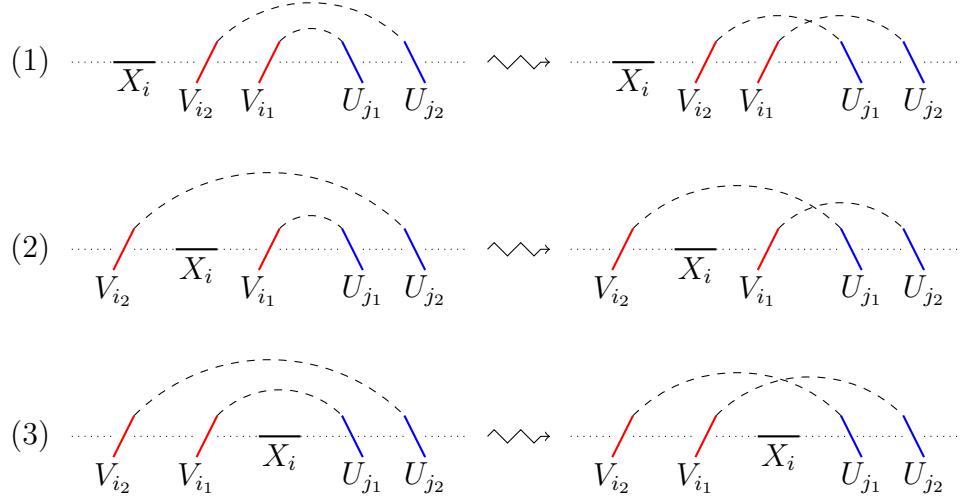


FIGURE 1. Relative positions of simple moves.

Proof of Theorem 9.1. By Theorem 4.3, we have to show

$$(c_1(\xi_i) \cup \text{Stab}_{\mathfrak{C}_-}(D), \text{Stab}_{\mathfrak{C}_+}(D'))_{\text{virt}} = \begin{cases} \iota_D^*(c_1(\xi_i)) & \text{if } D' = D, \\ \text{sgn}(D, D')h & \text{if } D' \in \text{SM}_{D,i}, \\ 0 & \text{otherwise.} \end{cases} \quad (9.9)$$

By Proposition 4.5, the virtual scalar product in (9.9) is a linear polynomial in the equivariant parameters. Hence, it suffices to prove the above equality in $H_{\mathbb{T}}^*(\text{pt})/(h^2)$. Suppose first that $D = D'$. As the partial orders $\preceq_+ = \preceq_{\mathfrak{C}_+}$ and $\preceq_- = \preceq_{\mathfrak{C}_-}$ defined in Subsection 4.1 are opposite, the support condition for stable basis elements implies that $\iota_T^*(\text{Stab}_{\mathfrak{C}_-}(D)) = 0$ for $T \not\preceq_- D$ and $\iota_T^*(\text{Stab}_{\mathfrak{C}_+}(D)) = 0$ for $T \prec_- D$. Thus, we have

$$\begin{aligned} (c_1(\xi_i) \cup \text{Stab}_{\mathfrak{C}_-}(D), \text{Stab}_{\mathfrak{C}_+}(D))_{\text{virt}} &= \sum_{D'' \in \mathcal{C}(\mathcal{D})^{\mathbb{T}}} \frac{\iota_{D''}^*(c_1(\xi_i) \cup \text{Stab}_{\mathfrak{C}_-}(D) \cup \text{Stab}_{\mathfrak{C}_+}(D))}{e_{\mathbb{T}}(T_{D''}\mathcal{C}(\mathcal{D}))} \\ &= \frac{\iota_D^*(c_1(\xi_i) \cup \text{Stab}_{\mathfrak{C}_-}(D) \cup \text{Stab}_{\mathfrak{C}_+}(D))}{e_{\mathbb{T}}(T_D\mathcal{C}(\mathcal{D}))}. \end{aligned} \quad (9.10)$$

Then, the normalization condition yields

$$(9.10) = \frac{\iota_D^*(c_1(\xi_i)) \cup e_{\mathbb{T}}(T_D\mathcal{C}(\mathcal{D})_{\mathfrak{C}_-}^-) \cup e_{\mathbb{T}}(T_D\mathcal{C}(\mathcal{D})_{\mathfrak{C}_+}^-)}{e_{\mathbb{T}}(T_D\mathcal{C}(\mathcal{D}))} = \iota_D^*(c_1(\xi_i)).$$

This proves (9.9) for $D = D'$. Next, we assume $D' \in \text{SM}_D$ and (V_{i_1}, U_{j_1}) be the right moving tie and (V_{i_2}, U_{j_2}) be the left moving tie of D . By Corollary 9.12, we have modulo h^2 :

$$\begin{aligned} (c_1(\xi_i) \cup \text{Stab}_{\mathfrak{C}_-}(D), \text{Stab}_{\mathfrak{C}_+}(D'))_{\text{virt}} &\equiv \frac{\iota_D^*(c_1(\xi_i) \cup \text{Stab}_{\mathfrak{C}_-}(D) \cup \text{Stab}_{\mathfrak{C}_+}(D'))}{e_{\mathbb{T}}(T_D\mathcal{C}(\mathcal{D}))} \\ &\quad + \frac{\iota_{D'}^*(c_1(\xi_i) \cup \text{Stab}_{\mathfrak{C}_-}(D) \cup \text{Stab}_{\mathfrak{C}_+}(D'))}{e_{\mathbb{T}}(T_{D'}\mathcal{C}(\mathcal{D}))}. \end{aligned} \quad (9.11)$$

Then, Corollary 9.11 and the normalization condition imply

$$\frac{\iota_D^*(c_1(\xi_i) \cup \text{Stab}_{\mathfrak{C}_-}(D) \cup \text{Stab}_{\mathfrak{C}_+}(D'))}{e_{\mathbb{T}}(T_D \mathcal{C}(\mathcal{D}))} \equiv \text{sgn}(D, D') h \frac{\iota_D^*(c_1(\xi_i))}{t_{j_2} - t_{j_1}} \mod h^2, \quad (9.12)$$

whereas Proposition 9.13 combined with the normalization condition gives

$$\frac{\iota_{D'}^*(c_1(\xi_i) \cup \text{Stab}_{\mathfrak{C}_-}(D) \cup \text{Stab}_{\mathfrak{C}_+}(D'))}{e_{\mathbb{T}}(T_{D'} \mathcal{C}(\mathcal{D}))} \equiv \text{sgn}(D, D') h \frac{\iota_{D'}^*(c_1(\xi_i))}{t_{j_1} - t_{j_2}} \mod h^2. \quad (9.13)$$

Inserting (9.12) and (9.13) in (9.11) yields

$$(9.11) \equiv \text{sgn}(D, D') h \frac{\iota_D^*(c_1(\xi_i))}{t_{j_2} - t_{j_1}} \mod h^2. \quad (9.14)$$

Now, Lemma 9.16 implies

$$(9.14) \equiv \begin{cases} \text{sgn}(D, D') h \mod h^2 & \text{if } D \in \text{SM}_{D,i}, \\ 0 \mod h^2 & \text{if } D \notin \text{SM}_{D,i}. \end{cases}$$

Thus, we proved (9.9) for $D' \in \text{SM}_D$. Finally, it remains to prove (9.9) for $D' \notin \text{SM}_D \cup \{D\}$. By Corollary 9.12, this assumption implies that h^2 divides all localization coefficients $\iota_T^*(\text{Stab}_{\mathfrak{C}_-}(D) \cup \text{Stab}_{\mathfrak{C}_+}(D'))$. Thus, we conclude

$$(c_1(\xi_i) \cup \text{Stab}_{\mathfrak{C}_-}(D), \text{Stab}_{\mathfrak{C}_+}(D'))_{\text{virt}} \equiv 0 \mod h^2$$

which proves (9.9) for $D' \notin \text{SM}_D \cup \{D\}$ and hence completes the proof of Theorem 9.1. \square

9.5. Proof of Proposition 9.10. Note that by Proposition 5.1, it suffices to show Proposition 9.10 for essential brane diagrams as defined in Subsection 5.1. Hence, we assume throughout this subsection that \mathcal{D} is essential.

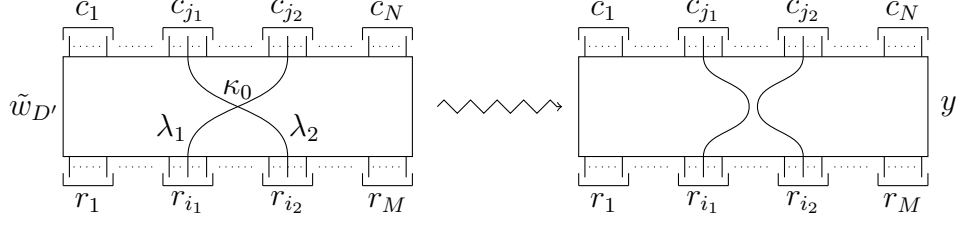
Our crucial tool is the following:

Lemma 9.17. *Let $D, D' \in \text{Tie}(\mathcal{D})$, $z \in w_{D'} S_{\mathbf{r}}$ and d_z a reduced diagram of shape (8.4). Suppose $K(d_z, w_D, 1) \neq \emptyset$, where $K(d_z, w_D, 1)$ is defined as in Proposition 8.10. Then, we have $D' \in \text{SM}_D \cup \{D\}$.*

Proof. Given $K \in K(d_z, w_D, 1)$ then, as d_z is of shape (8.4), we distinguish between the following two cases:

- (1) all crossings in K are contained in the boxes corresponding to $w_{0,c_1}, \dots, w_{0,c_N}$ and v_1, \dots, v_M ,
- (2) exactly one crossing $\kappa_0 \in K$ is contained in the box corresponding to $\tilde{w}_{D'}$ and the remaining crossings of K are contained in the boxes corresponding to $w_{0,c_1}, \dots, w_{0,c_N}$.

If (1) is satisfied then resolving all crossings contained in K from d_z still gives a permutation in $S_{\mathbf{c}} w_D S_{\mathbf{r}}$. Thus, we have $S_{\mathbf{c}} w_D S_{\mathbf{r}} = S_{\mathbf{c}} w_{D'} S_{\mathbf{r}}$ which implies $D = D'$ by Corollary 7.3. Assume now that (2) is satisfied. Let $d_{\tilde{w}_{D'}}$ be the reduced diagram for $\tilde{w}_{D'}$ contained in d_z . We denote by λ_1, λ_2 the strands in $d_{\tilde{w}_{D'}}$ that intersect in κ_0 and let $y \in S_n$ be the permutation that is obtained from $d_{\tilde{w}_{D'}}$ by resolving the crossing κ_0 . In pictures, y is obtained as follows:



Let f_1, f_2 resp. g_1, g_2 the starting resp. endpoints of λ_1, λ_2 in $d_{\tilde{w}_{D'}}$. As in the above picture, we assume $f_1 < f_2$. As $\tilde{w}_{D'}$ is the shortest element in $S_{\mathbf{c}}\tilde{w}_{D'}S_{\mathbf{r}}$ there exist $i_1 < i_2$ and $j_1 < j_2$ such that

$$R_{i_1-1} < f_1 \leq R_{i_1}, \quad R_{i_2-1} < f_2 \leq R_{i_2}, \quad C_{j_1-1} < g_2 \leq C_{j_1}, \quad C_{j_2-1} < g_1 \leq C_{j_2}.$$

Thus, we conclude

$$F_y(f_1) = F_{\tilde{w}_{D'}}(f_2) = j_1, \quad F_y(f_2) = F_{\tilde{w}_{D'}}(f_1) = j_2, \quad F_y(i) = F_{\tilde{w}_{D'}}(i), \quad \text{for } i \neq f_1, f_2. \quad (9.15)$$

Here, $F_y, F_{\tilde{w}_{D'}}$ are defined as in (7.2). By assumption, $y \in S_{\mathbf{c}}\tilde{w}_D$. Thus, we have $F_y = F_{\tilde{w}_D}$. Hence, by passing to the associated matrices of these double cosets, we deduce that (9.15) is equivalent to

$$\begin{aligned} M(D)_{i_1, j_1} &= M(D)_{i_2, j_2} = 1, & M(D')_{i_1, j_1} &= M(D')_{i_2, j_2} = 0, \\ M(D)_{i_1, j_2} &= M(D)_{i_1, j_2} = 0, & M(D')_{i_1, j_2} &= M(D')_{i_1, j_2} = 1, \end{aligned}$$

as well as $M(D)_{l, k} = M(D')_{l, k}$ for $(l, k) \notin \{(i_1, j_1), (i_1, j_2), (i_2, j_1), (i_2, j_2)\}$. Therefore, by Lemma 9.3, D' is obtained from D via a simple move. \square

The proof of Proposition 9.10 follows now from Proposition 8.10 and Lemma 9.17:

Proof of Proposition 9.10. We have to show that the localization coefficient $\iota_{D'}^*(\text{Stab}_{\mathbf{c}_-}(D))$ is divisible by h^2 for all $D' \notin \text{SM}_D \cup \{D\}$. Assume $\iota_{D'}^*(\text{Stab}_{\mathbf{c}_-}(D))$ is not divisible by h^2 for some $D' \notin \text{SM}_D \cup \{D\}$. By definition, this implies that $\iota_{D'}^*(\widetilde{\text{Stab}_{\mathbf{c}_-}(D)})$ is also not divisible by h^2 . Thus, by Proposition 8.10, there exists $z \in w_{D'}S_{\mathbf{r}}$ with reduced diagram d_z of shape (8.4) such that $K(d_z, w_D, 1) \neq \emptyset$. Hence, Lemma 9.17 yields $D' \in \text{SM}_D \cup \{D\}$ which contradicts our assumption $D' \notin \text{SM}_D \cup \{D\}$. \square

9.6. Proof of Proposition 9.13. Again, we can assume that \mathcal{D} is essential.

We need some further notation: let $D \in \text{Tie}(\mathcal{D})$ and $D' \in \text{SM}_D$ with right moving tie (V_{i_1}, U_{j_1}) and left moving tie (V_{i_2}, U_{j_2}) . Given $z \in w_{D'}S_{\mathbf{r}}$ with a reduced diagram d_z of shape (8.4) then there exist unique strands λ_1, λ_2 in d_z with starting points f_1, f_2 and endpoints g_1, g_2 such that

$$R_{i_1-1} < f_1 \leq R_{i_1}, \quad R_{i_2-1} < f_2 \leq R_{i_2}, \quad C_{j_1-1} < g_2 \leq C_{j_1}, \quad C_{j_2-1} < g_1 \leq C_{j_2}.$$

Let κ_0 denote the crossing of λ_1 and λ_2 .

To prove Proposition 9.13 we utilize the approximation formula of Proposition 8.10. To apply this formula appropriately, we use the following lemma:

Lemma 9.18. *Let $\tilde{y}_D \in S_n$ be as in (9.3) and $y_D = (w_{0,c_1} \times \dots \times w_{0,c_N})\tilde{y}_D$. Then, we have*

$$K(d_z, y_D, 1) = \begin{cases} \{\kappa_0\} & \text{if } z = w_{D'}, \\ \emptyset & \text{if } z \neq w_{D'}, \end{cases}$$

where $K(d_z, y_D, 1)$ is defined as in Proposition 8.10.

Proof. Let $z = w_{D'}v$ where $v \in S_r$ and suppose $K \in K(d_z, y_D, 1)$. As in Subsection 8.3, we denote by $K_U(d_z)$ the crossings in d_z corresponding to the boxes of $w_{0,c_1}, \dots, w_{0,c_N}$. By assumption $|K \setminus K_U(d_z)| \leq 1$. Thus, as z is fully separated, we have $K \setminus K_U(d_z) = \{\kappa_0\}$. By construction, resolving the crossing κ_0 from d_z gives a diagram for $y_D v$. Hence, Proposition 7.10 implies that $v = \text{id}$ and $K \cap K_U(d_z) = \emptyset$ which proves the lemma. \square

By combining Proposition 8.10 and Lemma 9.18, we obtain the following consequence:

Corollary 9.19. *We have that $\iota_{D'}^*(\widetilde{\text{Stab}}_{\mathfrak{e}_-}(D))$ is congruent modulo h^2 to*

$$\frac{\text{sgn}(D, D') \cdot h \cdot \left(\prod_{\alpha \in L'_{w_{D'}}} \Psi_{\mathcal{D}}(\alpha + h) \right) \cdot \left(\prod_{\kappa \in K_W(d_{w_{D'}}) \setminus \{\kappa_0\}} \Psi_{\mathcal{D}}(\text{wt}(\kappa)) \right)}{\prod_{\beta \in R_r} \Psi_{\mathcal{D}}(w_{D'} \cdot \beta)}.$$

Here, we used the notation from Proposition 8.10 and $K_W(d_{w_{D'}})$ is defined as in Subsection 8.3.

Proof. If we choose $w' = y_D$ in (8.5) then, by Lemma 9.18, the only set of crossings that contributes to (8.5) is $K(d_{w_{D'}}, y_D, 1) = \{\kappa_0\}$. Thus, by Proposition 8.10, $\iota_{D'}^*(\widetilde{\text{Stab}}_{\mathfrak{e}_-}(D))$ is congruent modulo h^2 to

$$\frac{(-1)^{\ell(y_D) + \ell(w_D)} \cdot h \cdot \left(\prod_{\alpha \in L'_{w_{D'}}} \Psi_{\mathcal{D}}(\alpha + h) \right) \cdot \left(\prod_{\kappa \in K_W(d_{w_{D'}}) \setminus \{\kappa_0\}} \Psi_{\mathcal{D}}(\text{wt}(\kappa)) \right)}{\prod_{\beta \in R_r} \Psi_{\mathcal{D}}(w_{D'} \cdot \beta)}.$$

By Proposition 9.8, $(-1)^{\ell(w_D) + \ell(y_D)} = \text{sgn}(D, D')$ which proves the corollary. \square

Proof of Proposition 9.13. By definition, we have

$$\frac{\iota_{D'}^*(\text{Stab}_{\mathfrak{e}_-}(D))}{e_{\mathbb{T}}(T_{D'}\mathcal{C}(\mathcal{D})_{\mathfrak{e}_-}^-)} = \frac{\iota_{D'}^*(\text{Stab}_{\mathfrak{e}_-}(D))}{\iota_{D'}^*(\text{Stab}_{\mathfrak{e}_-}(D'))} = \frac{\iota_{D'}^*(\widetilde{\text{Stab}}_{\mathfrak{e}_-}(D))}{\iota_{D'}^*(\widetilde{\text{Stab}}_{\mathfrak{e}_-}(D'))}. \quad (9.16)$$

Proposition 8.10 gives that $\iota_{D'}^*(\widetilde{\text{Stab}}_{\mathfrak{e}_-}(D'))$ is congruent modulo h to

$$\frac{\left(\prod_{\alpha \in L'_{w_{D'}}} \Psi_{\mathcal{D}}(\alpha) \right) \cdot \left(\prod_{\kappa \in K_W(d_{w_{D'}})} \Psi_{\mathcal{D}}(\text{wt}(\kappa)) \right)}{\prod_{\beta \in R_r} \Psi_{\mathcal{D}}(w_{D'} \cdot \beta)}. \quad (9.17)$$

Combining (9.17) and Corollary 9.19 then yields

$$(9.16) \equiv \frac{\text{sgn}(D, D') \cdot h}{\Psi_{\mathcal{D}}(\text{wt}(\kappa_0))} \equiv \frac{\text{sgn}(D, D') \cdot h}{t_{j_1} - t_{j_2}} \pmod{h^2},$$

which proves Proposition 9.13. \square

9.7. Chevalley–Monk formula for arbitrary chamber. Employing Theorem 5.3, generalizes the Chevalley–Monk formula for the antidominant chamber from Theorem 9.1 to any choice of chamber:

Theorem 9.20. *Let $\mathfrak{C} = z^{-1} \cdot \mathfrak{C}_-$ for $z \in S_N$, D be a tie diagram of \mathcal{D} and $i \in \{1, \dots, M\}$. Then, the following identity holds in $H_{\mathbb{T}}^*(\mathcal{C}(\mathcal{D}))_{\text{loc}}$:*

$$c_1(\xi_i) \cup \text{Stab}_{\mathfrak{C}}(D) = \iota_D^*(c_1(\xi_i)) \cdot \text{Stab}_{\mathfrak{C}}(D) + \sum_{D' \in \text{SM}_{D,z,i}} \text{sgn}_z(D, D') h \cdot \text{Stab}_{\mathfrak{C}}(D'),$$

where $\text{SM}_{D,z,i} = \{D' \in \text{Tie}(\mathcal{D}) \mid z.D' \in \text{SM}_{z.D,i}\}$ and $\text{sgn}_z(D, D') = \text{sgn}(z.D, z.D')$.

Let $\text{SM}_{D,z} = \bigcup_{i=1}^M \text{SM}_{D,z,i}$. If $D' \in \text{SM}_{D,z}$ then we say that D' is obtained from D via a z -twisted simple move.

Proof of Theorem 9.20. At first, note that (2.6) implies

$$\iota_T^*(\xi_i(\mathcal{D})) = w^{-1} \cdot (\iota_{w.T}^*(\xi_i(w.\mathcal{D}))), \quad \text{for all } w \in S_N \text{ and } T \in \text{Tie}(\mathcal{D}). \quad (9.18)$$

Employing (9.18) and Theorem 5.3 for a given $T \in \text{Tie}(\mathcal{D})$ yields

$$\iota_T^*(c_1(\xi(\mathcal{D}))) \cup \widetilde{\text{Stab}}_{\mathfrak{C}}(D) = z^{-1} \cdot (\iota_{z.T}^*(c_1(\xi_i(z.\mathcal{D}))) \cup \widetilde{\text{Stab}}_{\mathfrak{C}_-}(z.D)). \quad (9.19)$$

Then, Theorem 9.1 gives that (9.19) is equal to

$$z^{-1} \cdot \left(\iota_{z.T}^* \left(\iota_{z.D}^* (c_1(\xi_i(z.\mathcal{D}))) \cdot (\widetilde{\text{Stab}}_{\mathfrak{C}_-}(z.D)) \right) + \sum_{D' \in \text{SM}_{z.D,i}} \text{sgn}(z.D, D') h \cdot \widetilde{\text{Stab}}_{\mathfrak{C}_-}(D') \right). \quad (9.20)$$

Applying again (9.18) and Theorem 5.3 then gives

$$(9.20) = \iota_T^* \left(\iota_D^* (c_1(\xi_i(\mathcal{D}))) \cdot \text{Stab}_{\mathfrak{C}}(D) + \sum_{D' \in \text{SM}_{D,z,i}} \text{sgn}_z(D, D') h \cdot \text{Stab}_{\mathfrak{C}}(D') \right).$$

Therefore, we conclude Theorem 9.20 by the localization theorem. \square

10. CHEVALLEY–MONK FORMULAS IN THE GENERAL CASE

In the previous section, we proved Chevalley–Monk formulas for bow varieties with separated brane diagram (Theorem 9.20). Via Hanany–Witten transition, we finally deduce Chevalley–Monk formulas for bow varieties corresponding to arbitrary choices of brane diagram and chamber.

10.1. Simple moves for general brane diagrams. Fix a brane diagram \mathcal{D} . First, we generalize the notion of (twisted) simple moves:

Definition 10.1. For $D \in \text{Tie}(\mathcal{D})$, we define the *set of simple moves* SM_D as the set of all $D' \in \text{Tie}(\mathcal{D})$ such that there exist $1 \leq i_1 < i_2 \leq M$ and $1 \leq j_1 < j_2 \leq N$ satisfying

- (i) $M(D)_{i_1,j_1} = M(D)_{i_2,j_2} = 1$ and $M(D)_{i_1,j_2} = M(D)_{i_2,j_1} = 0$,
- (ii) $M(D')_{i_1,j_1} = M(D')_{i_2,j_2} = 0$ and $M(D')_{i_1,j_2} = M(D')_{i_2,j_1} = 1$,
- (iii) $M(D)_{l,k} = M(D')_{l,k}$ for all $(l,k) \notin \{(i_1,j_1), (i_2,j_1), (i_1,j_2), (i_2,j_2)\}$.

If $D' \in \text{SM}_D$ we say that D' is obtained from D via a simple move.

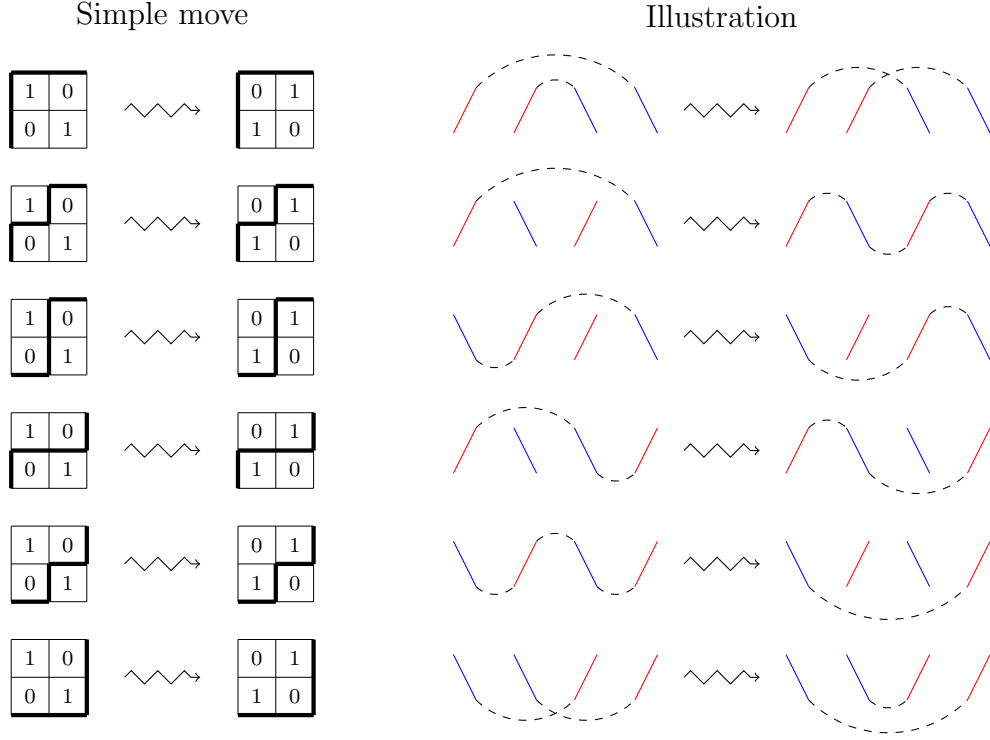
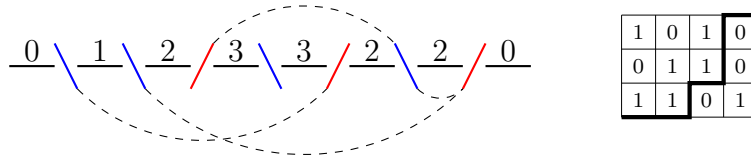


FIGURE 2. Illustration of simple moves for general brane diagrams.

Given additionally $i \in \{1, \dots, M\}$, we define the *set of simple move relative to i* $\text{SM}_{D,i}$ as the set of all tie diagrams D' of \mathcal{D} such that there exists $1 \leq i_1 \leq M - i + 1 \leq i_2 \leq M$ as well as $1 \leq j_1 < j_2 \leq N$ satisfying (i)-(iii).

The graphical illustration of simple moves depends on the position of the separating line relative to the respective 2×2 submatrix where the simple move is performed. The six possible cases are recorded in Figure 2.

Example 10.2. Let $\mathcal{D} = 0 \backslash 1 \backslash 2 / 3 \backslash 3 / 2 \backslash 2 / 0$. As tie diagram D we choose



The tie diagrams that are obtained from D via a simple moves have the following binary contingency tables:

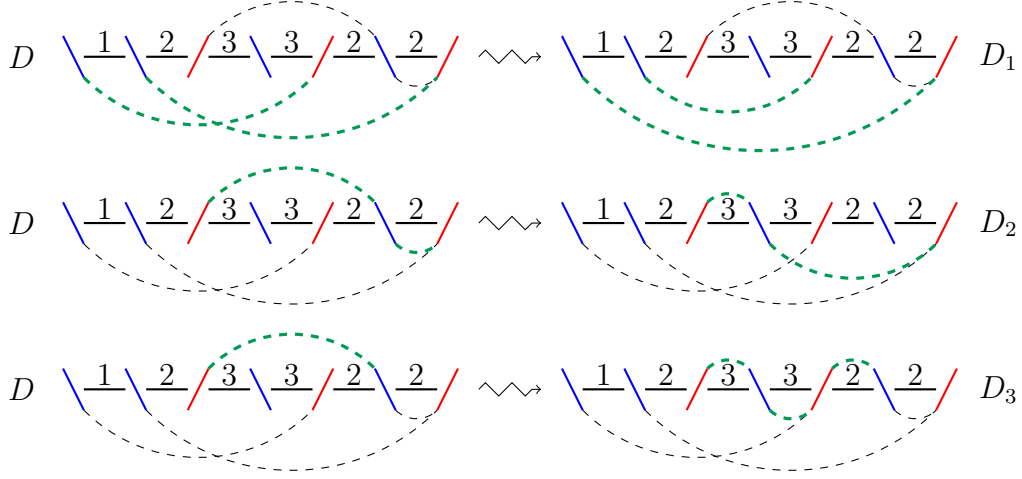
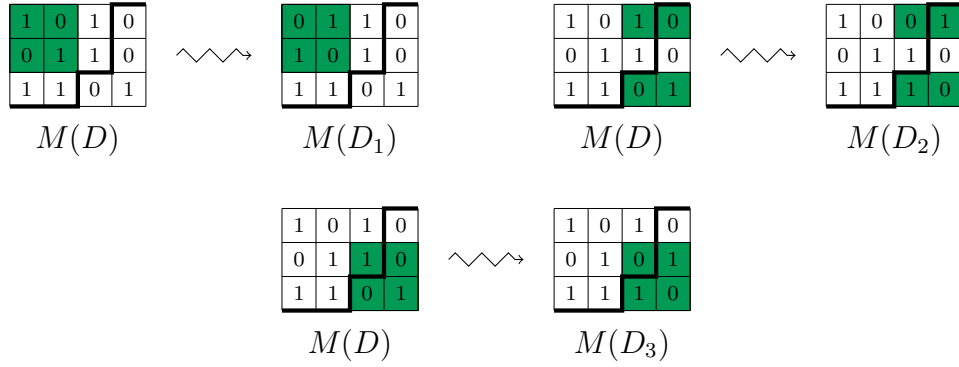


FIGURE 3. Example of simple moves for general brane diagrams.



We highlighted the submatrices that are involved in the respective simple moves. The corresponding illustrations of the simple moves in terms of tie diagrams are illustrated in Figure 3.

If $D' \in \text{SM}_D$ then the *sign of the simple move between D and D'* is defined as

$$\text{sgn}(D, D') := (-1)^{n_1+n_2}, \quad \text{where } n_1 = \sum_{l=j_1+1}^{j_2-1} M(D)_{i_1,l}, \quad n_2 = \sum_{l=j_1+1}^{j_2-1} M(D)_{i_2,l}.$$

The notion of twisted simple moves also generalizes as expected: for $z \in S_N$ we set

$$\text{SM}_{D,z} := \{D' \in \text{Tie}(\mathcal{D}) \mid z.D' \in \text{SM}_{z.D}\}$$

and

$$\text{SM}_{D,z,i} := \{D' \in \text{Tie}(\mathcal{D}) \mid z.D' \in \text{SM}_{z.D,i}\}, \quad \text{for } i = 1, \dots, M.$$

If $D' \in \text{SM}_{D,z}$, we say that D' is obtained from D via a z -twisted simple move. The corresponding *sign of the z -twisted simple move between D and D'* is defined as $\text{sgn}_z(D, D') := \text{sgn}(z.D, z.D')$.

By Lemma 9.3, the definitions of (z -twisted) simple moves and the corresponding signs agree with the previous definitions for separated brane diagrams.

Simple moves are well-behaved with respect to Hanany–Witten transition: let \tilde{D} be the brane diagram obtained via Hanany–Witten transition from \mathcal{D} by switching $U_{j_0} \in \mathbf{b}(\mathcal{D})$ and $V_{i_0} \in \mathbf{r}(\mathcal{D})$. Let $\Phi : \mathcal{C}(\mathcal{D}) \xrightarrow{\sim} \mathcal{C}(\tilde{\mathcal{D}})$ be the corresponding Hanany–Witten isomorphism (see Proposition 2.15) and let $\phi : \text{Tie}(\mathcal{D}) \xrightarrow{\sim} \text{Tie}(\tilde{\mathcal{D}})$ be the induced bijection.

Lemma 10.3. *Let $D, D' \in \text{Tie}(\mathcal{D})$, $z \in S_N$ and $i \in \{1, \dots, M\}$. Then, we have $D' \in \text{SM}_{D,i,z}$ if and only if $\phi(D') \in \text{SM}_{\phi(D),i,z}$.*

Proof. The proof is immediate from the fact that $M(D) = M(\phi(D))$ for all $D \in \text{Tie}(\mathcal{D})$, see Proposition 2.16. \square

10.2. Chevalley–Monk formula in the general case. We finally formulate and prove Chevalley–Monk formulas for bow varieties corresponding to not-necessarily separated brane diagrams.

We first set up some notation: given a brane diagram \mathcal{D} and $i \in \{1, \dots, M-1\}$ then we set

$$I(\mathcal{D}, i) := \{X \in \mathbf{h}(\mathcal{D}) \mid V_{i+1} \triangleleft X \triangleleft V_i\}.$$

In addition, we set

$$I(\mathcal{D}, 0) := \{X \in \mathbf{h}(\mathcal{D}) \mid V_1 \triangleleft X\} \quad \text{and} \quad I(\mathcal{D}, M) := \{X \in \mathbf{h}(\mathcal{D}) \mid X \triangleleft V_M\}.$$

For instance, let $\mathcal{D} = 0 \backslash 1 \backslash 2/3 \backslash 3/2 \backslash 2/0$ as in Example 10.2. Then, one can easily check that $I(\mathcal{D}, 0) = \{X_8\}$, $I(\mathcal{D}, 1) = \{X_7, X_6\}$, $I(\mathcal{D}, 2) = \{X_5, X_4\}$, $I(\mathcal{D}, 3) = \{X_3, X_2, X_1\}$.

Now, we finally state the general Chevalley–Monk formula:

Theorem 10.4. *Let $\mathfrak{C} = z^{-1} \cdot \mathfrak{C}_-$ for $z \in S_N$ and $i \in \{0, \dots, M+1\}$. Then, we have the following identity in $H_{\mathbb{T}}^*(\mathcal{C}(\mathcal{D}))_{\text{loc}}$:*

$$c_1(\xi_j) \cup \text{Stab}_{\mathfrak{C}}(D) = \iota_D^*(c_1(\xi_j)) \cdot \text{Stab}_{\mathfrak{C}}(D) + \sum_{D' \in \text{SM}_{D,z,i}} \text{sgn}_z(D, D') h \cdot \text{Stab}_{\mathfrak{C}}(D'),$$

for all $X_j \in I(\mathcal{D}, i)$ and $D \in \text{Tie}(\mathcal{D})$.

For the proof, we use the following notation: given a \mathbb{T} -equivariant vector bundle E on $\mathcal{C}(\mathcal{D})$, we denote by $C(\mathcal{D}, E) = C(\mathcal{D}, E)_{D,D'}$ the matrix with entries in $H_{\mathbb{T}}^*(\text{pt})_{\text{loc}}$ corresponding to the operator of multiplication with $c_1(E)$ on $H_{\mathbb{T}}^*(\mathcal{C}(\mathcal{D}))_{\text{loc}}$ with respect to the stable basis $(\text{Stab}_{\mathfrak{C}}(D))_{D \in \text{Tie}(\mathcal{D})}$.

We begin with the following lemma:

Lemma 10.5. *Let $U_{j_0} \in \mathbf{b}(\mathcal{D})$, $V_{i_0} \in \mathbf{r}(\mathcal{D})$, $\tilde{\mathcal{D}}$, Φ and ϕ be as in Lemma 10.3. Let $X_l = U_{j_0}^+$ and $D, D' \in \text{Tie}(\mathcal{D})$. Then, we have*

$$\varphi_{j_0}(C(\tilde{\mathcal{D}}, \tilde{\xi}_j)_{\phi(D), \phi(D')}) = C(\mathcal{D}, \xi_j)_{D,D'}, \quad \text{for } j \neq l$$

and

$$\varphi_{j_0}(C(\tilde{\mathcal{D}}, \tilde{\xi}_l)_{\phi(D), \phi(D')}) = C(\mathcal{D}, \xi_{l+1})_{D,D'} + C(\mathcal{D}, \xi_{l-1})_{D,D'} - C(\mathcal{D}, \xi_l)_{D,D'} + (t_{j_0} + h)\delta_{D,D'}.$$

Here, $\tilde{\xi}_i$ is the tautological bundle over $\mathcal{C}(\tilde{\mathcal{D}})$ corresponding to X_i , $\delta_{D,D'}$ is the Kronecker delta and $\varphi_{j_0} : \mathbb{Q}[t_1, \dots, t_N, h] \xrightarrow{\sim} \mathbb{Q}[t_1, \dots, t_N, h]$ is the $\mathbb{Q}[h]$ -algebra automorphism given by $t_{j_0} \mapsto t_{j_0} + h$ and $t_j \mapsto t_j$ for $j \neq j_0$.

Proof. Let $\Phi^*: H_{\mathbb{T}}^*(\mathcal{C}(\tilde{\mathcal{D}})) \xrightarrow{\sim} H_{\mathbb{T}}^*(\mathcal{C}(\mathcal{D}))$ be the induced ring isomorphism from Φ . By Proposition 4.6, we have $\Phi^*(\text{Stab}_{\mathfrak{C}}(\phi(T))) = \text{Stab}_{\mathfrak{C}}(T)$ for all $T \in \text{Tie}(\mathcal{D})$. Thus,

$$\varphi_{j_0}(\mathcal{C}(\tilde{\mathcal{D}}, \tilde{\xi}_i)_{\phi(D), \phi(D')}) = \mathcal{C}(\mathcal{D}, \Phi^*(\tilde{\xi}_i))_{D, D'}.$$

Hence, the lemma follows from Corollary 2.17. \square

Proof of Theorem 10.4. We prove the theorem via induction on the separatedness degree of \mathcal{D} . The case $\text{sdeg}(\mathcal{D}) = 0$ is exactly the statement of Theorem 9.20, so let us assume $\text{sdeg}(\mathcal{D}) > 0$. As in the proof of Theorem 9.1, the support condition for stable basis elements directly implies

$$\mathcal{C}(\mathcal{D}, \xi_j)_{D, D} = \iota_D^*(c_1(\xi_j)), \quad \text{for all } D \in \text{Tie}(\mathcal{D}).$$

Hence, it is left to show that $\mathcal{C}(\mathcal{D}, E)$ has the correct off-diagonal terms. As $\text{sdeg}(\mathcal{D}) > 0$, there exist $U_{j_0} \in \mathfrak{b}(\mathcal{D})$ and $V_{i_0} \in \mathfrak{r}(\mathcal{D})$ as in Lemma 10.5. In the following, we use the notation from Lemma 10.5. Let $D, D' \in \text{Tie}(\mathcal{D})$ with $D \neq D'$. Assume first that $X_j \neq U_{j_0}^+$. Note that in this case $X_j \in I(\tilde{\mathcal{D}}, i)$. Hence, the induction hypothesis gives

$$\mathcal{C}(\tilde{\mathcal{D}}, \tilde{\xi}_j)_{\phi(D), \phi(D')} = \begin{cases} \text{sgn}_z(\phi(D), \phi(D'))h & \text{if } \phi(D') \in \text{SM}_{\phi(D), i, z}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, Lemma 10.3 and Lemma 7.12 imply

$$\mathcal{C}(\mathcal{D}, \xi_j)_{D, D'} = \begin{cases} \text{sgn}_z(D, D')h & \text{if } D' \in \text{SM}_{D, i, z}, \\ 0 & \text{otherwise.} \end{cases}$$

So $\mathcal{C}(\mathcal{D}, \xi_j)$ has the correct off-diagonal terms. It remains to prove the case $X_j = U_{j_0}^+$. Note that in this case $i = i_0$. Since $X_{j+1} \in I(\mathcal{D}, i-1)$ and $X_j \in I(\tilde{\mathcal{D}}, i-1)$, the induction hypothesis and the previous case imply $\varphi_{j_0}(\mathcal{C}(\tilde{\mathcal{D}}, \tilde{\xi}_j)_{\phi(D), \phi(D')}) = \mathcal{C}(\mathcal{D}, \xi_{j+1})_{D, D'}$. Therefore, Lemma 10.3 and Lemma 7.12 again imply that $\mathcal{C}(\mathcal{D}, \xi_j)$ and $\mathcal{C}(\mathcal{D}, \xi_{j-1})$ have identical off-diagonal entries. By the first case, the latter are given by

$$\mathcal{C}(\mathcal{D}, \xi_{j-1})_{D, D'} = \begin{cases} \text{sgn}_z(D, D')h & \text{if } D' \in \text{SM}_{D, i, z}, \\ 0 & \text{otherwise,} \end{cases}$$

which completes the proof. \square

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